A TWO–ITEM TWO–WAREHOUSE PERIODIC REVIEW INVENTORY MODEL WITH TRANSSHIPMENT

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ABSTRACT

This thesis considers a two-item two-warehouse periodic review inventory model that allows transshipment between warehouses. Transshipment decision between warehouses is dependent on the time to the next order and the state of inventory in the other warehouse when a stock out occurs in a warehouse. The objective function considered is the total operating cost which comprises the variable ordering costs, the holding costs at the warehouses, the costs of transshipment between warehouses and the cost of emergency orders if transshipment is not possible. An infinite horizon dynamic programming model is used to develop the objective function. As the resulting optimization problem is a non-linear integer programming problem, we propose a heuristic to solve the problem. The proposed heuristic is a combination of Greedy heuristics and Lagrangian relaxation methods. The advantage of the Lagrangian method is its ability to provide a test for the quality of the solution. A series of numerical experiments performed not only illustrates the method proposed but also shows that optimality can be achieved using the proposed heuristic. Further, the method also provides the optimal instants when emergency orders will be preferred over transshipment.

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CHAPTER I

INTRODUCTION

The logistics business is going through a major shift in its characteristics since e-tailing has become a dominant form of order fulfillment. This research is motivated by what is happening in the retailing industry such as electronics, gifts and books that have added e-tailing as a demand fulfillment opportunity in addition to the traditional retailing businesses. The industry is characterised by a high volume, high variety and large customer base scenarios.

The total value of shipments in the US upsurged 5% in 2007 from the previous year. In the same period the value of shipments generated through e-commerce increased by 12.1%. There was a surge in e-commerce activity especially in the area of B-to-B segment as suggested by E-Stats of US census bureau (2009)[1]. A BBC report (2007)[2] on the UK's e-fulfillment market states that the online shopping in the UK increased by 33.4% to reach a figure of £10.9bn in 2006. Online retailing is expected to be worth about £28.1bn (8.9% of the UK's total retail sales) by 2011. As the numbers above show, the usage of e-fulfillment by the consumers has increased. The proliferation of e-commerce has been so immense and the consequent advantages are so great that governments and firms offer incentives to promote e-commerce transactions.

1.1 E-fulfillment and its features

In spite of the advantages of e-commerce, e-fulfillment presents a new set of challenges vis-a-vis the traditional methods. Xu [41] presents a comparison of the features of

[1] http://www.census.gov/econ/estats/2007/2007reportfinal.pdf
[2] http://news.bbc.co.uk/2/hi/business/6690397.stm

online retailing with those of the traditional method. The most striking comparison is that e-tailing involves a larger catalog and size of operations. Clearly, the e-fulfillment system is an assembled-to-order system where a set of items ordered is assembled together to customize every customer order. Empirical research findings (see [21]) show that in e-tailing, the time committed for delivery is the yardstick of the consumer's trust. Hence, reliable and efficient logistics is paramount in e-tailing. Agatz et al [1] also discuss the different features of e-fulfillment. They highlight the manufacturer's freedom in pricing decisions which naturally leads to revenue management and the need for pooling to offer a better catalog. Further, challenges exist when this is performed at remarkable speed and be cost effective. According to Lummus and Vokurka [25] E-fulfillment includes the traditional features and characteristics of delivery processes but the main difference lies in the interaction though the interface of internet. This has made the e-tailer to look at various alternatives to the method of delivery. These include existing distribution center, a dedicated e-fulfillment center, third party logistics, direct shipments, etc.

1.2 Inventory Models for E-Fulfillment

As explained in the previous section, inventory models to facilitate e-fulfillment need a unique set of features. The inventory system for e-fulfillment requires a multi-item inventory model with the additional fulfillment capabilities in event of a stockout such as usage of transshipments or emergency shipments and an ability to deal with a large volume of products.

Since lower cost and timely delivery of the product is the indicator of the customer perception in e-fulfillment, a range of options for the fastest but cost effective fulfillment has to be considered. The model developed by us incorporates the usage of emergency shipment methods as an option for fulfillment. Further, transshipments from depots in the same echelon are also proposed and the model recommends a decision of transshipment or emergency shipment upon stockout at a depot.The utility of transshipment to this inventory model is explained in the section below.

An e-fulfillment system would have to handle a larger number of items in the catalogue rather than in a traditional system. But to understand the issues better, we study a two-item two-depot model. Then, we discuss the multi-item case as an easy extension of the two-item case. Thus the models developed incorporate a set of attributes so as to address the unique features of e-fulfillment systems.

1.3 Performing E-logistics and E-Fulfillment

The advent of business over Internet has impacted much the order fulfillment function. There are two main areas where the impact has been felt the most. One of them is the process of the placement of the order itself which has in fact become much more efficient for the customer. However, for the supplier the order fulfillment process can be expensive. The other is the use of the Internet in increasing the efficiency of the fulfillment process by way of easy access for information the Internet offers and further easy management of a large amount of data obtained (see Gimenez and Lourenco (2004) [15])

The players in the E-fulfillment market also play in the conventional order fulfillment market. The industry becomes complicated with traditional retail firms moving into the e-space. So an understanding of multi-channel distribution is needed to essentially understand E-fulfillment (see Agatz et al. [1]). The authors conceive the supply chain to have four stages: (i) sales - interface of customer demands, (ii) delivery - physical movement of products, (iii) warehousing - storing functions (iv) purchasing - ordering and fulfillment functions.

As with any traditional business function, the e-logistics and e-fulfillment operations too can be performed in-house or outsourced or drop-shipped. Bayles [7] compares these options and provides the advantages and disadvantages of each. If warehousing is not the core competency of the company, outsourcing that function is a better option. Outsourcing also ensures that the company can use the third party's existing infrastructure to its advantage. Third party service providers are of two kinds – nonasset-based and asset-based. The core competency of asset based service providers is the routing and physical vehicles in comparison to the nonasset-based providers. However, the major disadvantage of using third party logistics is the involvement of a new entity and hence the loss of control. Further options for fulfillment exists with drop-shipping in which, the process involves the three steps of a product sale, making the purchase order and the manufacturer's fulfillment of the order directly. However, drop-shipping too has its disadvantages such as returns handling and after sales service.

Another area that is relatively new is that of Fourth party logistics (4PL). Fourth party logistics is seen as a provider who owns the intellectual capital to provide logistics for a part or the whole supply chain of the firm. They may use a third party logistics provider to supply service to customers. Warrilow and Beaumont [36] provide a good discussion on the usage of 4PLs against the more traditional 3PLs.

1.4 Transshipment in E-fulfillment

A common practice to optimize costs in inventory management is the usage of transshipment (movement of item in the same echelon). As explained below, transshipment helps in a number of ways during e-fulfillment. The utility of transshipment to e-fulfillment models are pooling of inventories leading to improvement in timely service which is an indicator of the standard of service, increase of items in catalog, decrease of split orders and easy handling of final assembly and product returns.

From the foregoing it is clear that the inventory system to cater to the need of e-fulfillment requires a multi-warehouse multi-item inventory model with options for timely delivery of customer demand at the lowest cost to the customer. The timely delivery of the customer demand is dependent on the stock availability of the items demanded at the warehouses. So, when a stockout occurs at a warehouse, a lateral shipment of that item can be made from other warehouses or an emergency shipment has to be made from the supplier or the central warehouse or backorder be made to fulfill the demand.

There exists a lot of literature on the usage of transshipment in inventory models. However, the study of transshipment with respect to periodic inventory model is limited and multi-item inventory model are few as mentioned in the literature review below. Further, system approaches, which has been explained in detail in the literature review, has been rarely used. Archibald et al [4] derived formulas for the single item two-depot periodic inventory model with transshipment and emergency shipments (see Figure 1). The authors also proposed a heuristic to find the solution of two-item two-warehouse model. But, as the authors themselves claim, the heuristic proposed is not successful for certain instances. Hence, in this thesis, we embark on proposing a heuristic that can solve all the instances in the more general case of multi-item two-warehouse periodic inventory problem.

1.5 Layout of the thesis

The research objectives of this thesis are to introduce new and more realistic assumptions than what has been considered in the literature for the problem described. Further, we intend to study the system with respect to costs contributors and suggest approximation heuristics to obtain solutions with the introduction of capacity constraints. Finally, we would improve the model to multiple inventory cases and provide approximate heuristics.

To this end, the present chapter explained above the features of e-fulfillment, our research goals and the significance of the study. In what follows, Chapter 2 reviews

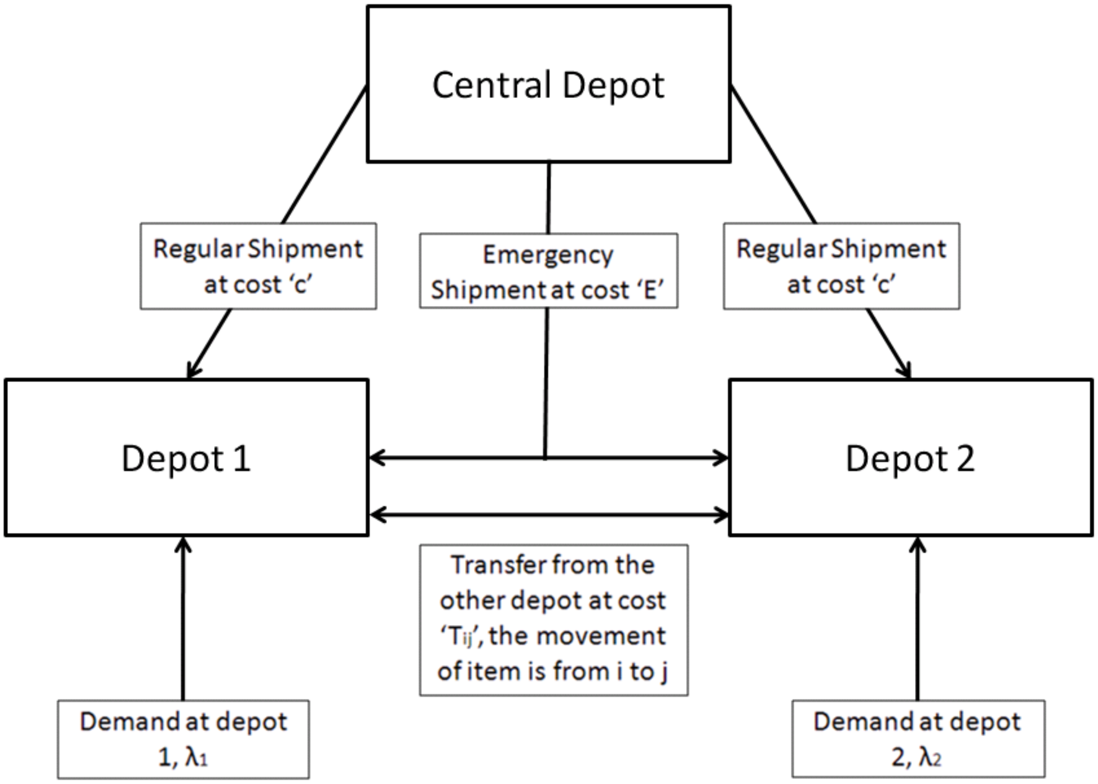

Figure 1: A single item two-warehouse model

the state of art in the area of inventory models with transshipment and identifies the research gaps in the area of periodic inventory with transshipment. The research methodology subsection provides methods to solve the integer non-linear problem. In chapter 3, we present the formulation of the two-item two-warehouse periodic review inventory model with transshipment. Chapter 4 presents the different search procedures proposed in this thesis to obtain the solutions. The procedure consists of three heuristics used in succession. The heuristics are compared in chapter 5. The highlights on the utility of the methods are also provided. The insights and analysis of the solutions obtained from the procedures are explained in chapter 6. Chapter 7 demonstates through numerical examples the applicability of our model to multi-item cases. Chapter 8 provides a summary of our findings as the conclusion.

CHAPTER II

LITERATURE REVIEW

2.1 Introduction

From an academic standpoint, continuous review models in multi-item inventory system have been fairly dealt with. However, the literature on periodic inventory system for multiple items and transshipment has been minimal. From a business standpoint, a common strategy used in inventory management to be cost effective is lateral transshipment. As demands arrive, a situation may arise wherein one retail outlet might face stock-outs while the other faces only a modest demand. So warehouses use lateral shipments to satisfy these orders rather than resorting to emergency shipments. Since transshipment contributes to inventory pooling, service levels in these systems are higher. This improves the economic viability, more specifically in e-fulfillment scenario where the volume is varied and speed is of importance.

This thesis, hence, attempts to propose heuristics for achieving solutions for a two-warehouse, two-product case with periodic inventory policy and transshipment and then extend the formulation to a generalized multi-item inventory case. The section on literature review which follows below, gives a summary of the research works in this area.

2.2 Literature on Inventory with transshipments

In an E-fulfillment scenario, the order fulfillment involves multiple depots and multiple items. Wong et al. [38] have provided a detailed list of inventory models with transshipment. They further explain the two approaches considered in the study of transshipment in an inventory system. The two approaches are item approach and system approach. Item approach considers only one item at a time and is hence single-item problems compounded. The second approach as proposed by Sherbrooke [32] is the system approach in which the stocking decisions are taken with all item in simultaneous consideration. The second approach proves to be more cost effective. However, most of the literature in transshipment uses an item approach. A few of the papers which have used system approach include Archibald et al. [4] and Wong et al [38] . The predominant literature available for item approaches have been is reviewed below and the literature using system approach has been tabulated in Table 1

The literature in inventory models with transshipment can be categorized based on the number of items in the model, number of echelons, number of depots and the inventory policy used such as periodic inventory, base stock etc. Analysis of inventory models with transshipment started with single-item inventory systems, mostly based on repairable items models. Gross [16] considered transshipment before demand arrival and modeled a policy to optimise transshipment cost. Krishnan and Rao [22] developed an inventory model with reactive transshipments and derived optimal solutions for two-locations. An initial research in this field is the METRIC model developed by Sherbrooke [31]. Many extensions of this model have also been proposed. Lee [24] developed an approximate model considering transshipment between identical bases. Axsater [5] applied the same to transshipment between non-identical bases.

Robinson [29] studied an inventory system using transshipment as recourse for stockout after demand realisation for basestock ordering policy and proved the optimality of the basestock policy for the conditions of only two locations or identical cost parameters. Tagaras and Cohen [33] used the periodic inventory system to study the benefits of complete pooling. Partial pooling methods were studied in comparison to complete pooling methods and were always found to be inferior to complete pooling. Evers [11] studied a (Q, r) system for ensuring the benefit of emergency transshipment by a pooling approach within the same echelon. The paper concluded that the benefit lies in the consolidation of lead-times. Needham and Evers [27] used a (Q, r) policy to study the interactions between different cost factor and transshipments by simulation methods. The simulation results indicated that the decision for transshipment primarily depended on the cost of a stockout. Evers [12] examined the effectiveness of different pooling methods. Alfredsson and Verrijdt [2] modeled a two-echelon inventory system with the assumption that the demands can be satisfied by an emergency shipment or a lateral shipment or a further emergency shipment from an infinite capacity plant in case local and central warehouses were out of stock. Herer and Rashit [19] modeled an inventory system characterized by joint replenishment and derived an optimal inventory policy for it. Tagaras [34] studied the importance of pooling through simulations of different pooling scenarios for warehouses. Grahovac and Chakravarthy [17] studied both centralized and decentralized systems with the base stock policy assuming independent stochastic lead times. The paper found that savings for sharing of inventory would not go together with decrease in inventory. Evers [13] studied a (Q, r) model and provided two heuristics to determine the situation in which stock transfers are cost effective given the inventory levels. Rudi et al [30] considered situations for which the local depots exercise control over the transshipment prices and inventory and maximize their profits. Axsater [6] studied the (Q, r) system with lateral shipments in a single direction. Further an approximate system for the problem has been proposed and a simulation study is used to study the substitution effect in inventory systems. Xu, Evers and Fu [40] studied service level for a two location inventory model with transshipment assuming a (Q, r) system for inventory and a control factor over the amount of transshipment moving out of the depot. Chiang and Monahan [9] analysed the dual channel strategy. The demand intheir model was both from an online and a retail store with the fulfillment effected

Literature	Continous/ Periodic	Policy	No. of depot	No. of items	Pooling of Inventory
Archibald et al [4]	Periodic	Order-up-to	2	m	Decision based
Wong et al [37]	Continous	$(S-1, S)$	n	m	Complete pooling
Wong et al [38]	Continous	$(S-1, S)$	2	m	Complete pooling
This thesis	Periodic	Order-up-to	2	m	Decision based

Table 1: The literature using system approach for inventory model with transshipment

from both the retail location and a manufacturing base. The paper identified situations in which the dual strategy would be the better strategy. Kutanoglu [23] studied an inventory system with transshipment for time-based service levels and further performed the cost analysis for the model. Hu, Watson and Schneider [20] developed approximate solutions for order quantities using dynamic programming approach for (s, S) inventory models with backordering and centralised ordering.

Archibald et al [4] modelled a transshipment system using (s, S) policy and developed expressions for order quantities for two-depot two-item case. A heuristic for the two-depot, multiple-item inventory problem was also proposed. However the decision variables were the holding costs where appropriate holding costs are calculated to allocate the inventory to fill the capacity to the full. Heuristics are also provided by Archibald [3]. Wong et al [38] have proposed a Lagrangian relaxation based heuristic to solve multi-item continuous review inventory system. Bounds for the solutions were obtained too.

As indicated above periodic inventory systems with transshipment have been studied only by a select few. The system approach of considering all the item jointly for arriving at the order levels has been sparingly used too. The methods proposed also are heuristics. The issue of capacity of the warehouse, though present in the formulation of Archibald et al [4] does not represent reality. Expressions are derived for the single item periodic inventory case with transshipments and emergency shipments. The paper has further explained a heuristic for the two-depot multi-item inventory system. The solution finds a feasible solution by varying the holding costs in each iteration. The values of holding costs are reduced in each iteration through a bisection method until a feasible solution is obtained. However, in real-life situations, holding cost would be based on the amount of inventory and interest rates. Hence, the premise on changing holding cost independent of the amount of inventory is debatable. Further, the sensitivity analysis of the results with respect to costs shows a variation of about 35%. The procedure does not provide solutions in every situation and fails if a set of holding costs can not be obtained to fill the capacity of the depots. Hence, we propose to study a two-warehouse, two-product periodic inventory model with transshipment and capacity constraints. These extensions would enhance the applicability to the business case.

2.3 Research Methodology

The two-warehouse two-item inventory problem is a constrained non-linear problem (NLP) in the discrete space. Wu [39] reviewed four traditional approaches to solve the integer NLP. One method is the conversion of the problem to a constrained 0-1 NLP. The process involves rewriting the problem as a binary problem. However, the rewritten problem is more complicated due to the addition of many new variables which makes tractability a problem. The present methods for solving are however limited. Second method is the usage of penalty functions. Penalty is a weight added for any violation of constraints. Through the use of penalty, the problem is converted into an unconstrained problem. Greedy searches are employed to move in the solution space to find better solutions with each iterations. As with any greedy procedure, the procedure would provide only local minimum. The third approach is the search for direct solutions by either rejection of non-feasible solutions in the solution space or by usage of randomised searches. Finally, in the Lagrangian Relaxation approach in which the primal problem is converted into a dual problem using Lagrangian multiplier for each the constraints. Hence the problem is converted into an unconstraint optimisation problem. However, the solution procedure become very complex with increase in the nonlinearity of the primal. While finding a solution to our problem, we have used the Lagrangian procedure to test the quality of the initial solution and observe whether optimality is possible.

After performing a comparative study of the methods described above, we have chosen to use the Lagrangian relaxation approach for our model due to the following reasons:

1. Lagrangian relaxation approaches the problem with a systems perspective rather than an item perspective. Hence achievement and study of optimality is possible.

2. Procedures for finding bounds and improvement of the bounds with Lagrangian methods are available in the literature.

3. Since in our problem the objective function is an implicit function, the constrained $0-1$ NLP cannot be applied due to the complexity involved.

4. The Global search methods and metahueristics cannot guarantee optimality and the bounds are difficult to ascertain.

The utility of Lagrangian relaxation method to solve optimization problems was initially provided by Everett [10]. Fisher [14] gives an example based guide on the usage of Lagrangian relaxation to optimization problems. Porteus [28] has presented an extensive literature on the advantages of Lagrangian formulations, methods and utility.

CHAPTER III

TWO-WAREHOUSE TWO-ITEM PERIODIC REVIEW INVENTORY MODEL

3.1 Introduction

We consider a two-item two-warehouse periodic review inventory system served by a central source with the two warehouses acting as storage and selling locations. Demand for the items at the warehouses are assumed to be Poisson and the warehouse capacities are finite. An example of such a system is a company selling their products online. The two warehouses are located in such a way so as to service the two areas around them. In fact, this model is based on a real life scenario described in Miller et al. [26] where an online retailer of pet food tries to minimize the split delivery that occurs consequent to the transshipments between warehouses when one warehouse runs out of stock. This is due to the fact that the customers may resent receiving split deliveries of their orders. They develop heuristics to find appropriate bundles of products that each of the warehouses should carry with the objective of minimizing the total number of split orders. In their model the authors assume a bulk factor which depends on the order quantity and safety stock for that item. Thus, for their model, the order quantity and safety stock are parameters. Motivated by this real life observation, we aim to extend the work of Miller et al in finding the optimal order levels for items given the bundles of items the warehouses carry. To this end, we first consider a two-item two capacitated warehouses model. The items could be bundles of products as considered in the above work. We assume that both the warehouses carry both the bundles.

In what follows, we first explain our model together with the assumptions and the notations. Then we develop the cost function and the associated constrained non-linear discrete optimization problem. In developing the cost function we have borrowed the results from Archibald et al [4] which are also surveyed below for easy reference.

3.2 Model

We consider an online retailer with two capacitated warehouses. Each of these warehouses serve an area each. For simplicity, we assume that the warehouses carry only two items. In a subsequent chapter, we will show how the model can be extended to multiple items. The retailer uses a periodic review inventory policy. Without loss of generality, we assume that the length of a period is 1. The demands for each of the items occur at the warehouses according to a Poisson process. The Poisson processes are independent. Replenishment orders for both items are made jointly by each of the warehouses to a central warehouse. The central warehouse also replenishes the stocks jointly. If a demand occurs during a stock out at a warehouse, that demand can be satisfied by either a transshipment from the other depot or by an emergency order to the central source which results in extra costs. These are all common assumptions used in the literature. In the case of transshipment, the decision to accept the transshipment request by the other warehouse is dependent on the time still remaining until the next period (or the next order). This is also a decision variable for both the warehouses. In particular, let τ_i^k(with depot k having i units in stock at the time of a transshipment request), denote the time to the next order, the maximum time threshold beyond which a transshipment request from the other warehouse will not be entertained and hence the other warehouse will need to use emergency order to fulfill an unsatisfied demand. Besides, we also assume as in Archibald et al. [4] that the unsold inventory at the end of the period can be returned to the central warehouse for a full refund and then a new order is placed and received at the warehouses at the start of the next period.

Without loss of generality we assume that the SKUs are in units of pallets and hence their bulk factors are the same. Consequently, the storage space required by any SKU in the depot is the same.

Further, we consider instantaneous order and delivery of the item, i.e the lead time is zero. The ordering of stocks and its instantaneous delivery occurs in the start of the review period for all the items. The review period is also assumed to be the same for both the items and hence all orders and receipts of orders occur simultaneously and jointly at the same time.

We mention that as our model is an extension of Archibald et al. [4], our approach also closely follows the authors but our solution methodology is different from theirs. For ease of reference, we first provide the basic functions we borrow from Archibald et al. [4] and then derive the functions for our model.

We note that there are two decision problems in this model. The first is the decision to choose between an emergency order or transshipment at an instant within a review period when a demand occurs at a warehouse for a stocked out item. The second problem is the reordering decision at a review epoch. As in Archibald et al. [4], for the first problem we use a finite horizon continuous time Markov decision process while for the second we use an infinite horizon discounted Markov decision process. The underlying stochastic process is the inventory level process at the warehouses given by $\{(\mathbf{S}_1(t), \mathbf{S}_2(t)), t \geq 0\}$ with the state space given by $E = \{(\mathbf{s}_1, \mathbf{s}_2)\}$ with the vectors $\mathbf{s}_k = (s_{k1}, s_{k2})^T$ and $0 \leq s_{kj} \leq m_k$ for $k, j = 1, 2$. Specifically, we use the notation as given in Table 2.

A moment of reflection on the value function for the finite horizon Markov Decision process (MDP) will reveal the complexity involved in considering the events that occur in the evolution of the inventory level stochastic process. Hence, as a first level of

Symbol	Description
$\lambda_{k,j}$	Demand rate at warehouse $k, k = 1, 2$ for item $j, j = 1, 2$
m_k	Maximum storage capacity of warehouse $k, k = 1, 2$.
β	Discount factor
V	Infinite horizon discounted total cost (the objective function)
c	Cost of regular order per unit
$s_{k,j}$	Inventory at the start of any period of item j, $j = 1, 2$ in warehouse $k, k = 1, 2$
$W_1(s_{1j}, s_{2j})$	Minimum expected total cost per period for satisfying the demand for item j by transshipment or emergency orders
τ_{ij}^k	The threshold time until the next period for accepting a transshipment request at warehouse k holding an inventory of i units of item j at the instant of the transshipment request (a decision variable)
$f(\lambda, n, t)$	$=e^{-\lambda t} \frac{(\lambda t)^n}{n!}$
$F(\lambda, n, t)$	$=\sum_{i=0}^{n} e^{-\lambda t} \frac{(\lambda t)^i}{i!}$

Table 2: List of notation.

approximation, we propose to consider the two-item two-warehouse model to be two separate one-item two-warehouse models. The two separate models are inter-related through the capacity constraints. This now considerably simplifies the problem. Now, to develop the value function for this model, we first survey below the results of Archibald et al. [4] pertaining to the single item two-warehouse case.

3.2.1 Single item two-warehouse model of Archibald et al (1997)

In this section, we confine ourselves to a typical single item, say j held in two warehouses. We have the following results:

Lemma 1 (Archibald et al. (1997)) *Let, for $j = 1, 2$, $w_t^E(s_{1j}, s_{2j})$ and $w_t^T(s_{1j}, s_{2j})$ denote the minimum expected total costs until the next review epoch given that the time to the next review epoch is t when the system is in state (s_{1j}, s_{2j}) and there is an unmet demand at one of the warehouses for item j, which is satisfied by a transfer and* *an emergency order respectively. Then*

$$\begin{aligned} w_t^E(s_{1j},0) = E + \int_0^t \lambda_{2j} f(\lambda_{2j}, s_{2j}, u) &\left\{ \sum_{i_{1j}}^{s_{1j}} f(\lambda_{1j}, i_{1j}, u) w_{t-u}(s_{1j} - i_{1j}, 0) \right. \\ &\left. + \sum_{i_{1j}=s_{1j}+1}^{\infty} f(\lambda_{1j}, i_{1j}, u) \left((i_{1j} - s_{1j})E + w_{t-u}(0,0)\right) \right\} du \\ &+ f(\lambda_{2j}, 0, t) \left\{ \sum_{i_{1j}}^{s_{1j}} f(\lambda_{1j}, i_{1j}, u) W_0(s_{1j} - i_{1j}, 0) \right. \\ &\left. + \sum_{i_{1j}=s_{1j}+1}^{\infty} f(\lambda_{1j}, i_{1j}, u) \left((i_{1j} - s_{1j})E + W_0(0,0)\right) \right\} du \end{aligned}$$

and

$$w_t^T(s_{1j},0) = T_{1,2} - E + w_t^E(s_{1j} - 1, 0) \tag{1}$$

for $0 < s_{1j} \le m_1$. *In the above equations, the functions* $w_t(s_{1j},0)$ *and* $w_t(s_{1j},0)$ *are given as*

$$w_t(s_{1j},0) = \min\{w_t^E(s_{1j},0),\ w_t^T(s_{1j},0)\} \text{ for } 0 < s_{1j} \le m_1$$

and

$$w_t(0,s_{2j}) = \min\{w_t^E(0,s_{2j}), w_t^T(0,s_{2j})\} \text{ for } 0 < s_{2j} \le m_2.$$

Further,

$$W_0(s_{1j}, s_{2j}) = h_{1j}s_{1j} + h_{2j}s_{2j} - c_{1j}s_{1j} - c_{2j}s_{2j}.$$

Proof. *Usual conditional probability arguments on the next demand at warehouse 2, yields the above result.* ■

Theorem 2 (Archibald et al. (1997)) *For the finite horizon Markov decision process, the optimal value function satisfies the following: For* $j = 1, 2$, *the minimum expected* *total cost per period to satisfy the demands is*

$$W_1(s_{1j}, s_{2j}) = \int_0^1 \lambda_{1j} f(\lambda_{1j}, s_{1j}, t) \sum_{i_{2j}=0}^{s_{2j}} f(\lambda_{2j}, s_{2j}, t) w_{1-t}(0, s_{2j} - i_{2j}) dt \qquad (2)$$
$$+ \int_0^1 \lambda_{2j} f(\lambda_{2j}, s_{2j}, t) \sum_{i_{1j}=0}^{s_{1j}} f(\lambda_{1j}, s_{1j}, t) w_{1-t}(s_{1j} - i_{1j}, 0) dt$$
$$+ \sum_{i_{1j}=0}^{s_{1j}} \sum_{i_{2j}=0}^{s_{2j}} f(\lambda_{1j}, s_{1j}, t) f(\lambda_{2j}, s_{2j}, t) W_0(s_{1j} - i_{1j}, s_{2j} - i_{2j})$$

and

$$w_t(0,0) = E[1 + \lambda_{1j} + \lambda_{2j})t] + W_0(0,0) \qquad (3)$$

Proof. Usual probabilistic conditioning arguments on the instant of the occurrence of the first unmet demand, the above theorem can easily be proved. See also Archibald et al. [4]. ■

We are now ready to derive the expected discounted cost for the infinite horizon problem. We recall our assumption that we return the unsold items just prior to the start of the next period at no cost and with full refund for the returned items. We then make the order for the next period. Since, there is no fixed order cost in the model and the variable order cost is linear, the ordering decision at the start of any period is independent of the stock level just before the review. Hence, it is enough to consider only one state of the system at a review epoch. For convenience, as in Archibald et al., we take this state to be $(0,0)$ for each item $j = 1, 2$. Consequently, at any ordering instant, the system will not have any items in the warehouse. So, the decision problem is to know how many units to have at the start of any period. As per our definition, if we decide to have s_{kj} units of item j in warehouse k, with $k, j = 1, 2$, then the minimum expected total cost of satisfying the demands during the next period is $W_1(s_{1j}, s_{2j})$ given by (2). We now have the following theorem:

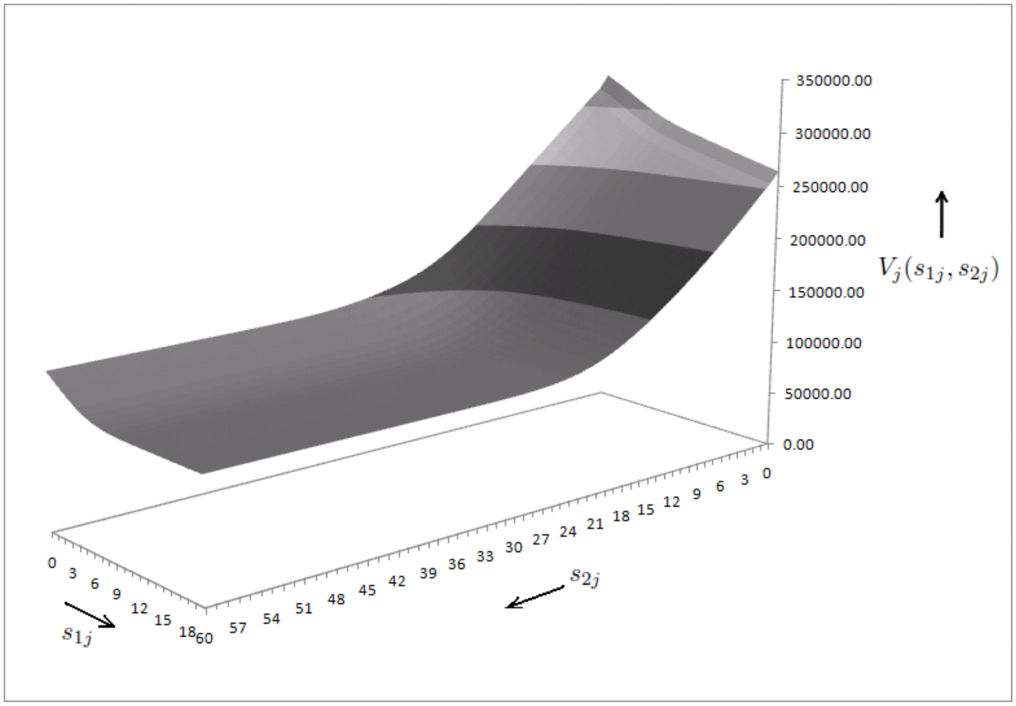

Figure 2: A realiziation of V

Theorem 3 (Archibald et al. (1997)) *If $V_j(0,0)$ is the infinite horizon minimum expected total discounted cost for item j in the two warehouses, then the optimal value function satisfies the following optimality equation.*

$$V_j(0,0) = \min \ \{cs_{1j} + cs_{2j} + \beta(W_1(s_{1j}, s_{2j}) + V_j(0,0)\} \tag{4}$$

where β is the discounting factor for future costs.

Proof. Proof is omitted as it is straightforward. ■

We note that the finite horizon MDP is a single state MDP and hence there is only one decision to make, viz the order up to levels. This decision is clearly the order up to levels (S_{1j}, S_{2j}) that minimize the RHS of (4). It is also known ([4]) that the functions involved are convex with respect to the inventory levels and sub-modular (see Topkis, [35]) in time and inventory level variables. For an illustration see Figure 2. Thus, we have the following theorem that characterizes the optimal policy:

Theorem 4 (Archibald et al (1997)) *For $j = 1, 2$, there exist nonnegative integers S_{1j} and S_{2j} such that the optimal reorder policy is to order up to level S_{1j} at warehouse 1 and to order up to level S_{2j} at warehouse 2.*

Using value iteration, Archibald et al. also prove the following structural results.

Theorem 5 (Archibald et al. (1997)) *For $j = 1, 2$, there exist real values $\tau^1_{1j} \leq \tau^1_{2j} \leq \cdots \leq \tau^1_{m_1 j}$ such that the minimising action in state $(s_{1j}, 0)$ when there is an unmet demand at warehouse 2 for item j and t time units to go until the next review epoch is to transfer an item from warehouse 1 to warehouse 2 if $t < \tau^1_{ij}$ and to place an emergency order otherwise. Similar result exists for the other warehouse.*

3.3 Our optimization problem

We now consider the two-item two-warehouse model. Using the same notation as before, the infinite horizon decision problem is:

Problem 1 (P0) *For two-item $(j = 1, 2)$ two-warehouse $(k = 1, 2)$ capacitated model, the decision problem is*

$$\min \sum_{j=1}^{2} V_j(s_{1j}, s_{2j}) \tag{5}$$

subject to

$$s_{11} + s_{12} \leq m_1$$

$$s_{21} + s_{22} \leq m_2$$

where m_1, m_2 are the capacities of warehouses 1 and 2 respectively and $V_j(s_{1j}, s_{2j})$ are given by 4.

The solution of the problem is the vector of order up to levels, $\mathbf{S} = (S_{11}, S_{12}, S_{21}, S_{22})$ which minimizes the total cost of the problem for the given capacity constraints.

It is clear that the problem is a non-linear integer programming problem with linear constraints. As pointed out earlier, Archibald et al. [4] have discussed this problem in passing in their paper where the main discussion was on the single item two-depot problem. They have provided an algorithm for solving this problem, based on the observations that (i) the warehouses needed to be filled to the full in the optimal solution (which they claim to have observed in a real life setting) and (ii) that the optimal solution does not depend on the holding cost rates. Hence, they have developed their algorithm through a search for the optimal holding cost rates that would fill the two warehouses to the full.

Their algorithm is only sketchy and they also mention that their algorithm may fail in certain cases. Hence, we embark in this thesis to finding an algorithm that is applicable to all situations and also an algorithm that would not fail. In this quest, we have chosen the Lagrangian relaxation approach together with heuristics for improvement if Lagrangian relaxation does not provide the optimal solution. The overall approach is supported by an initial greedy heuristic to speed up the process.

3.4 Lagrangian Relaxation

In this section, we describe our Lagrangian relaxation approach (see Porteus [28]). The choice of this approach is mainly based on the observation that the objective function and the constraints of problem **P0** are separable in terms of the items, i.e the problem actually splits into two single-item two-warehouse problem for which there is an exact algorithm proposed by Archibald et al [4]. In our approach, we use what are called Lagrangian multipliers Λ_j for each warehouse j which can be interpreted as the price for using the capacity in warehouse j. The procedure then tries to choose the best values for these parameters that would result in the best use of the capacities. To this end, we use a subgradient method. We first formulate the Lagrange relaxation problem below:

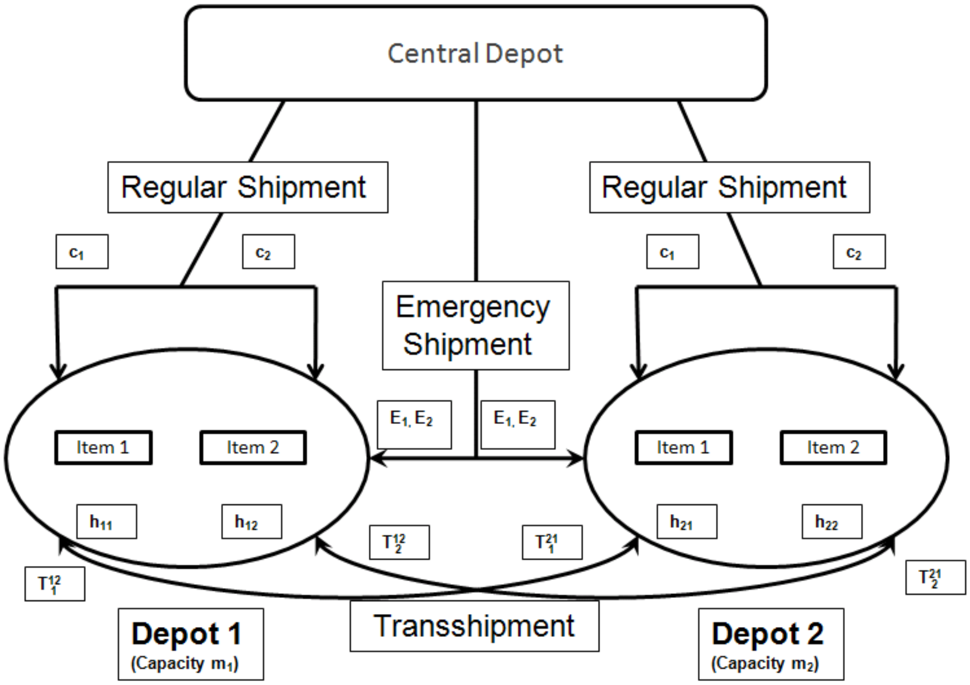

Figure 3: The two-warehouse two-item periodic inventory system

3.4.1 The Lagrangian relaxation problem

Let the vector $\mathbf{\Lambda} = (\Lambda_1, \Lambda_2)^T \epsilon \Re^2$ with $\Lambda_j \geq 0,\ j = 1, 2$ be the Lagrange multipliers. Now, by relaxing the capacity constraints in problem **P0** we obtain the following relaxed problem:

$$
\begin{aligned}
\min \sum_{j=1}^{2} V_j(s_{1j}, s_{2j}) + \Lambda_1(\sum_{j=1}^{2} s_{1j} - m_1) + \Lambda_2(\sum_{j=1}^{2} s_{2j} - m_2) \\
= \min \sum_{j=1}^{2} (V_j(s_{1j}, s_{2j}) + \Lambda_1 s_{1j} + \Lambda_2 s_{2j}) + \Lambda_1 m_1 + \Lambda_2 m_2 \\
= \min \sum_{j=1}^{2} F(s_{1j}, s_{2j}) + \Lambda_1 m_1 + \Lambda_2 m_2 \qquad (6)
\end{aligned}
$$

where $F(s_{1j}, s_{2j}) = \sum_{j=1}^{2} (V_j(s_{1j}, s_{2j}) + \Lambda_1 s_{1j} + \Lambda_2 s_{2j})$

Let $\mathbf{S} = (S_{11}, S_{12}, S_{21}, S_{22})$ be a solution to problem **P0** with the associated total cost given by $C(\mathbf{S})$. Let C^* be the optimal cost corresponding to optimal solution $\mathbf{S}^*$

for problem **P0**. Let $C^1_\Lambda(\mathbf{S})$ be the optimal cost of problem **P1**, for given set of Λs. Let $M_k = S_{k1} + S_{k2}$ be the capacity used in warehouse k for the ordering policy given by $\mathbf{S}$. We now have the following result (see Wong et al [38] for similar results for continuous review system with waiting time constraints):

Property 1 *From the formulation as given in problem **P1** we discern the following properties:*

(i) $C^* \geq C^1_\Lambda(\mathbf{S})$ for all $(\Lambda_1, \Lambda_2) \geq (0, 0)$.

(ii) $C^* \geq \max_\Lambda C^1_\Lambda(\mathbf{S})$

(iii) If for some $(\Lambda_1, \Lambda_2) \geq (0, 0)$ the optimal solution for problem **P1** is $\mathbf{S}^{1*}$ and $M_k \leq m_k$, $k = 1, 2$ then $\mathbf{S}^{1*}$ is feasible for problem **P0** and $C(\mathbf{S}^{1*}) - C^* \leq \Lambda_1(m_1 - M_1) + \Lambda_2(m_2 - M_2)$.

(iv) If for some $(\Lambda_1, \Lambda_2) \geq (0, 0)$ the optimal solution for problem **P1** is $\mathbf{S}^{1*}$ and for $k = 1, 2$, $M_k = m_k$, if $\Lambda_k > 0$; $M_k \leq m_k$ if $\Lambda_k = 0$, then $\mathbf{S}^{1*}$ is the optimal ordering policy for problem **P0.**

The proof is as follows.

(i) The result easily follows from the observation that any optimal solution to problem **P0** is a feasible solution for problem **P1** for any given $(\Lambda_1, \Lambda_2) \geq (0, 0)$. In turn, any feasible solution to problem **P1** should yield an objective value that is more than or equal to its optimal objective value.

(ii) The above result actually implies this result.

(iii) The first part is the consequence of definition of problem **P0** while the second part follows from (i).

(iv) This follows from (iii). One can refer to Everett [10] from which also the result follows.

The property above is useful in our search for the optimal solution to problem **P0**. First, we note that (i) above provides us with a lower bound for the optimal objective function value of problem **P0**. The next property (ii) indeed provides us with the best such lower bound. Further, from (iii) we have an upper bound for the gap between the objective function value for any feasible solution to **P0** and its optimal objective value. The final result indicates that the relaxed solution can be optimal to problem **P0** and if so the capacity of a warehouse will be fully utilized when the corresponding multiplier is positive and not fully utilized when the multiplier is zero.

The above property helps us develop an algorithm for finding the optimal solution. First, we note that we need an optimal solution to the relaxed problem for given Lagrange multipliers to get a lower bound. Then, we need to get the tightest lower bound for which we should find the best Lagrange multipliers. The approaches for finding these are presented below.

3.4.1.1 Optimal solution to the relaxed problem

It is clear that the Lagrangian relaxation problem **P1** is separable in the items and so the optimal solution to **P1** is got by solving the separated problems for items 1 and 2 using the algorithm of Archibald et al. [4] surveyed above. Thus, for a given set of values for (Λ_1, Λ_2) by solving the separated problems, we would obtain a lower bound for our original problem. The next task is to obtain the tightest lower bound and the associated best Lagrange multipliers $(\Lambda_1^*, \Lambda_2^*)$ for $C_\Lambda^1(\mathbf{S})$. But as the problem is a non-linear integer programming problem, it is not differentiable and so we cannot apply methods like steepest ascent. For such situations, the method usually employed is the subgradient optimization method (see Bazaara et al.[8]) which could be considered similar to steepest ascent method with the subgradient based direction replacing the gradient direction. We refer the readers to Fisher [14] for an excellent explanation on this procedure. We employ this procedure to our problem to find the optimal values of (Λ_1, Λ_2), a brief description of which is provided below.

Subgradient optimization The procedure involves updating (Λ_1, Λ_2) at each iteration using the subgradient direction calculated in that iteration. If at the n-th iteration, $(\Lambda_1^n, \Lambda_2^n)$ are the Lagrangian multipliers and if M_k^n be the total capacity used in warehouse k, the subgradient direction at iteration n is given by

$$\gamma_k^n = M_k^n - m_k^n. \tag{7}$$

The Lagrange multipliers are updated as follows:

$$\Lambda_k^{n+1} = \max(0, \Lambda_k^n t^n) \tag{8}$$

In the above, t^n is the step size which also needs updating at every iteration. The most commonly used updating procedure is the one proposed and justified by Held et al. [18]. The updating formula is

$$t^n = s^n \frac{C_{\Lambda^k} - \widehat{C}}{(\gamma_1^n)^2 + (\gamma_2^n)^2} \tag{9}$$

where $\widehat{C}$ is the best known upper bound for problem **P0** and s^n is a scalar between 0 and 2. If after a specified number of iterations, there is no improvement in the value of the objective function, the step size is updated by halving the value of t^n.

For this subgradient procedure, we need to initialize (Λ_1, Λ_2). Usually, $(0, 0)$ is chosen as the initial values for the Λs but a more efficient procedure to choose the initial values for (Λ_1, Λ_2) is proposed by Wong et al. [38]. The algorithm first finds three possible initial values for (Λ_1, Λ_2) and then chooses the best among the three.

Algorithm *The best initial Lambdas*

($*$ To get the best initial Lambdas $*$)

1. First point: Set $\Lambda_1 = 0$. Find the smallest value of Λ_2 for which the capacity constraints of problem P0 are satisfied
2. Second point: Set $\Lambda_2 = 0$. Find the smallest value of Λ_1 for which the capacity constraints of problem P0 are satisfied
3. Thrid point: Set $\Lambda_1 = \Lambda_2$. Find the smallest value of both Λ_1 and Λ_2 for which the capacity constraints of problem P0 are satisfied
4. Of the three above, choose the one with the largest objective function value for Problem P1 as the initial set of Lambdas for the subgradient procedure.
5. End

We also note that the optimal solution corresponding to the selected set of lambdas is a feasible solution for problem **P0.** This now provides an initial value for the upper bound $\widehat{C}$ that is required in the subgradient procedure.

Having set the background for solving to optimality our model proposed in this chapter, we now describe our experience in developing the solution procedure for solving our model in the next chapter.

CHAPTER IV

PROCEDURE FOR SOLVING OUR MODEL

4.1 Introduction

As described in the previous chapter, our optimization problem which is a non-linear integer programming problem can only be solved numerically. To this end, we described in the last chapter the Lagrangian relaxation based optimization with the use of subgradient procedure. We ran the procedure for some example problems, verifying the solution through the use of brute force. Our experience revealed that the convergence of the procedure and also the quality of the final solution depended very much on the initial value chosen for $\mathbf{S}$, the order levels. The usual choice of $\mathbf{S} = (0, 0, 0, 0)$ appeared to be not a good choice for the procedure. Hence, we needed to choose a good initial feasible $\mathbf{S}$. For this purpose, we have developed a greedy heuristic which by ignoring the capacity constraints uses $\mathbf{S} = (m_1, m_1, m_2, m_2)$. This is obviously an infeasible solution to find iteratively a good initial feasible solution. One other noteworthy observation is that in many of the example problems we solved, this initial greedy heuristic itself converges to the optimal solution but we needed either the brute force method or the Lagrangian relaxation approach to actually verify that the initial greedy heuristic indeed gave the optimal solution.

The next step was to use the Lagrangian relaxation approach supported by the subgradient optimization procedure to update (Λ_1, Λ_2). As we explained in the previous chapter, we also needed a good initial set for (Λ_1, Λ_2) for which we resorted to the procedure proposed by Wong et al. [38]. It is known that the Lagrangian relaxation approach may not always lead to the optimal solution to the original problem. We had many instances when it could not converge to the optimal solution. Hence, we needed a procedure to improve the solution provided by Lagrangian relaxation approach. A local neighbourhood procedure is used to improve the solution. Thus, the general procedure to solve our optimization problem is as follows:

Algorithm *The Main Procedure to solve Problem P0*

1. Use the greedy heuristic to find an initial feasible solution
2. Use the Lagrangian relaxation approach supported by the subgradient procedure to test whether the initial solution is optimal. If not, find iteratively a better solution. If optimality cannot be reached then go to the next step.
3. Use the Improvement heuristic to find a better solution to the one found in the above step.
4. End

Finally we compare the solution got from the above procedure to the solution obtained using brute force to evaluate the performance of the procedure. It is heartening to note that our procedure indeed performed better. We now describe each of the steps in our main procedure given above.

4.2 Initial Greedy Heuristic

As mentioned above, our initial greedy heuristic starts with an infeasible solution, ignoring the capacity constraints. The technique then removes one unit of either product from the current solution and calculates the difference in the value of cost term $V_j(s_{1j}, s_{2j})$ before and after the removal. The item that produces the least of the differences is removed from the current solution. This removal procedure is repeated until a feasible solution is reached. It should be noted that the i-th unit of product j is a candidate for removal only if the $(i-1)$-th unit of product j had already been removed. That is, at any iteration only one unit is removed. This heuristic is based on the observation that the function $V_j(s_{1j}, s_{2j})$ is convex with respect to (s_{1j}, s_{2j})

and so $\triangle V_j(s_{1j}, s_{2j})$ increases with increase in either s_{1j} or s_{2j}. Hence removal of one unit is certainly cheaper than removal of the subsequent unit.

4.2.1 Algorithm for the initial greedy heuristic

The initial greedy heuristic is as follows

Algorithm *Initial Greedy Heuristic*

(∗ To get an initial solution ∗)

1. $(s_{11}, s_{12}, s_{21}, s_{22}) \leftarrow (m_1, m_1, m_2, m_2)$
2. $S_1 \leftarrow s_{11} + s_{12}$ and $S_2 \leftarrow s_{21} + s_{22}$
3. **repeat**
4. **if** $s_{11} = 0$ **then** $s_{12} \leftarrow s_{12} - 1$. **if** $s_{12} = 0$ **then** $s_{11} \leftarrow s_{11} - 1$. **else** $d_1 \leftarrow V_1(s_{11} - 1, s_{21}) - V_1(s_{11}, s_{21})$, $d_2 \leftarrow V_2(s_{12} - 1, s_{22}) - V_2(s_{12}, s_{22})$. **if** $d_1 < d_2$, **then** $s_{11} \leftarrow s_{11} - 1$ **else** $s_{12} \leftarrow s_{12} - 1$
5. $S_1 \leftarrow s_{11} + s_{12}$
6. **until** $S_1 \leq m_1$
7. **repeat**
8. **if** $s_{21} = 0$ **then** $s_{22} \leftarrow s_{22} - 1$. **if** $s_{22} = 0$ **then** $s_{21} \leftarrow s_{21} - 1$. **else** $d_3 \leftarrow V_1(s_{11}, s_{21} - 2) - V_1(s_{11}, s_{21})$, $d_4 \leftarrow V_2(s_{12}, s_{22} - 1) - V_2(s_{12}, s_{22})$. **if** $d_3 < d_4$, **then** $s_{21} \leftarrow s_{21} - 1$ **else** $s_{22} \leftarrow s_{22} - 1$
9. $S_2 \leftarrow s_{21} + s_{22}$
10. **until** $S_2 \leq m_2$
11. End

4.2.2 Characteristics of the initial greedy heuristic

It is very clear that the heuristic will provide the optimal solution, if the problem is unconstrained. In the capacity constrained case, the method would follow the steepest descent till the inventories satisfy the capacity constraints. For discrete convex optimization case such as this problem, greedy procedure would not guarantee global optimality since the search space is discrete. Hence, we use the greedy heuristic to obtain the initial feasible solution without no concern for its optimality. However, the Lagrangian relaxation procedure can validate its optimality in all cases. We now describe the Lagrangian relaxation method.

4.3 Lagrangian Relaxation Method

We fist recall the Lagrangian relaxation problem (**P1**).

$$\begin{aligned} \min \sum_{j=1}^{2} V_j(s_{1j}, s_{2j}) + \Lambda_1(\sum_{j=1}^{2} s_{1j} - m_1) + \Lambda_2(\sum_{j=1}^{2} s_{2j} - m_2) \\ = \min \sum_{j=1}^{2} (V_j(s_{1j}, s_{2j}) + \Lambda_1 s_{1j} + \Lambda_2 s_{2j}) + \Lambda_1 m_1 + \Lambda_2 m_2 \\ = \min \sum_{j=1}^{2} F(s_{1j}, s_{2j}) + \Lambda_1 m_1 + \Lambda_2 m_2 \end{aligned} \tag{10}$$

where $F(s_{1j}, s_{2j}) = \sum_{j=1}^{2} (V_j(s_{1j}, s_{2j}) + \Lambda_1 s_{1j} + \Lambda_2 s_{2j})$.

Exploiting separability, we use the following Lagrangian relaxation algorithm. We also highlight that separability implies that the number of items need not be limited to two but can be easily extended to more than two items.

4.3.1 Algorithm of the Lagrangian Method

The following is the algorithm for the Lagrangian Heuristic.

Algorithm *Lagrangian Heuristic*

(∗ To get an improved solution ∗)

1. Get the initial solution **S** and its associated cost C from Algorithm *Initial Greedy Heuristic*
2. Get the initial values for Λ_1 **and** Λ_2 from the Lambda heuristic explained in the next section.
3. Solve the problem **P1** for the given set of Λ_1 **and** Λ_2

4. **if** the solution obtained in the previous step is feasible and lower than C **then** update **S** and its cost C_Λ with the current solution.
5. **if** $\mid C - C_\Lambda \mid \sim 0$ **then** End.
6. **else if S** is the same as any solution obtained in previous iterations, then the scalar $s \leftarrow s/2$.
7. Update the values of Λ_1, Λ_2 as follows and go to step 3

   - $\gamma_1 \leftarrow s_{11} + s_{12} - m_1$
   - $\gamma_2 \leftarrow s_{21} + s_{22} - m_2$
   - $t_n \leftarrow \frac{s(C_\Lambda - C)}{(\gamma_1)^2 + (\gamma_2)^2}$
   - $\Lambda_{1,n} \leftarrow \max(0, \Lambda_{1,n-1} - t_n\gamma_1)$
   - $\Lambda_{2,n} \leftarrow \max(0, \Lambda_{2,n-1} - t_n\gamma_2)$

8. End

4.3.2 Improtance of obtaining better solutions from Lagrangian Heuristics

4.3.2.1 Importance of a good initial feasible solution

A series of initial experiments using the algorithm were performed assuming the initial solution to be $\mathbf{S} = (0, 0, 0, 0)$. It was observed that in very many instances, the Lagrangian relaxation approach was not able to improve after a certain number of iterations. Hence there arose a need for a better starting solution. Subsequent runs with the starting solutions provided by the greedy heuristic converged to the optimal solution or at least to a near optimal solution.

4.3.2.2 Importance of a good set of initial Lagrangian Multipliers

As pointed out in the last chapter, the Lagrangian relaxation approach also depended on a good set of Lagrangian multipliers. To ascertain this, examples were executed with Lagrangian multipliers taking values in the set $\{1, 5, 50, 500, 5000\}$. Table 4 presents the instances when the algorithm converged to the optimal solution.

Value of Λ	Example 1	Example 2	Example 3
0	no	yes	no
1	no	yes	no
5	no	yes	no
50	yes	yes	yes
500	yes	yes	yes
5000	no	no	no

Table 3: Instances when optimal solutions were obtained when starting with different Λ

In the Table 3 we note that in about 50% of the cases it did not converge to the optimal solution. Thus, there is a need for a good starting solution. We have already mentioned in the last chapter that we use the three point approach advocated by Wong et al. [38].

Next, we describe the improvement heuristics that we use.

4.4 Improvement Heuristics

In spite of using good starting values for the initial solution and for the Lagrangian multipliers, we found instances when the procedure did not converge to the optimal solution. To improve this and to also explore the feasible region for a better solution, we propose a neighborhood search. From the value of $V_j(s_{1j}, s_{2j})$ obtained as the current upper bound in the Lagrangian relaxation procedure we identify the eight neighbors of (s_{1j}, s_{2j}) for product $j = 1, 2$ as shown in Figure 4. Combinations of solutions from the two sets are checked for satisfying the capacity constraints, discarding those combinations that violate these constraints. The neighbor yielding the minimum cost and which is also feasible, replaces the current upper bound. The process is repeated till the current upper bound solution did not change. Thus, we have the following algorithm for the improvement heuristic.

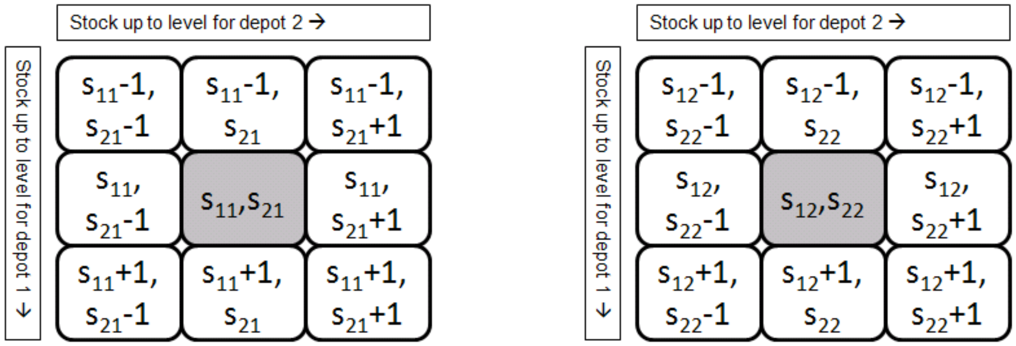

Figure 4: Graphic depiction of the improvement heuristic with the darker square representing the present solution and the lighter squares representing the neighborhood

4.4.1 Algorithm of Improvement Heuristics

Algorithm *Improvement Heuristic*

(∗ To get an improved solution ∗)

1. Get the initial solution **S** and its associated cost C from Algorithm *Lagrangian Heuristic*
2. Find the values of $V(S') = min(V_1(s_{11} + i_{11}, s_{21} + i_{21}) + V_2(s_{21} + i_{12}, s_{22} + i_{22}))$ where $i_{11}, i_{21}, i_{12}, i_{22} \varepsilon \{-1, 0, 1\}$.
3. **if** $V(S) \neq C$ **then** go to 2 **else** stop.
4. End.

4.5 A Numerical Experiment

We provide below an example from the numerical experiments we conducted. First, we give in Tables 4 and 5 the values of the parameters we used in the experiment.

The results of the experiments are displayed in Table 6. Brute force method was used just to check whether the solution obtained using the algorithms were optimal. We see that for this example, the initial heuristic itself provided the optimal solution. We find the Lagrangian relaxation approach also yielded the same. We verify this

Parameter	Units	Values
Capacity of depot 1 (m_1)	SKUs	40
Capacity of depot 2 (m_2)	SKUs	60
Z value		2.58
Arrival rate ratio for item 1:p_1,(100-p_1)	%	50%,50%
Arrival rate ratio for item 2:p_2,(100-p_2)	%	25%,75%
Holding cost interest rate (h)	\$/SKU-yr	0.03

Table 4: Input parameters for the numerical examples

Parameter	Units	Item1	Item2
Production cost of items	\$/SKU	20	20
Emergency cost	\$/SKU	50	50
Transshipment cost from depot 1 to 2	\$/SKU	40	40
Transshipment cost from depot 2 to 1	\$/SKU	40	40

Table 5: Item-wise cost parameters for the numerical examples

using the brute force method.

Method	s_{11}	s_{12}	s_{21}	s_{22}
Initial Greedy Heuristics	20	20	17	43
Lagrangian Heuristics	20	20	17	43
Brute Force	20	20	17	43

Table 6: Example output

CHAPTER V

COMPARISON OF HEURISTICS

5.1 Introduction

The problem at hand is a non-linear integer programming problem. The Lagrangian Heuristic provided a method by which the optimality can be achieved in most cases with the bounds provided in all cases. However these methods work better when provided with a good set of initial solution. Hence we used the greedy heuristic to provide initial solutions to start the Lagrangian procedure. Further an Improvement heuristic was added as a method subsequent to the Lagrangian procedure to better the output of the Lagrangian procedure. Our experience reveals that each of the procedures have inherent utilities as explained below.

The initial heuristic provides us with the first feasible solution. The cost of feasible solution would be either equal to or greater than that of the optimal solution. Hence the feasible solution provides the upper bound for the succeeding Lagrangian Heuristic. Lagrangian Procedure works better if the initial solution is nearer to the optimal solution.

The Lagrangian Heuristic is the only procedure in this paper which uses the systems approach to solve the problem rather than the unit approach. Hence the procedure obtains the solution considering both the depots together whereas the initial heuristic considered the depots independently. Secondly, Lagrangian Heuristic is the procedure that finds the bound and checks for the optimality of the solution. The Lagrangian procedure validates the optimality of the initial greedy heuristic. The solution obtained in the greedy heuristic is a feasible solution. The Lagrangian heuristic checks the optimality of the solution by comparing the solution with the bounds obtained in the Lagrangian Heuristic. The Lagrangian procedure ends if the solution equals the bounds.

The Improvement procedure is used only in cases where the Lagrangian relaxation fails. The Lagrangian heuristic gives us a scope for improvement because the solution is dependent on the inputs such as the initial solution provided, the values of Lagrangian multipliers used in the first iteration etc. If the procedure has not reached bounds with the solutions generated by the Lagrangian heuristics, the improvement method would find a better local minimum. The brute force method can alone verify if the solution is the global minimum in situations where the optimality is not achieved by the Lagrangian Heuristics.

5.2 Superiority of the Lagrangian Heuristic

As described in the sections above, the heuristics do not provide optimal solutions with certainty, though in all the cases we would be able to identify whether the solution is optimal or not. However, the Lagrangian heuristic would be the best heuristic to provide the optimal solution vis-a-vis the initial greedy heuristic.

The heuristics have been compared based on the procedure, stopping time, utility and convergence to optimality (see Table 7). With respect to the procedures, the initial and improvement heuristics handle unit inventory per iteration whereas Lagrangian relaxation handles multiple inventories in iteration. The improvement method is the quickest process as regards to time taken for the procedure to stop. The check for optimality is provided by the Lagrangian heuristic.

The most important utility of the Lagrangian heuristic is its use of the systems approach whereas the other heuristics resort to the item approach as the search procedure. Thus, the Lagrangian heuristic is oriented towards simultaneous joint optimisation as against optimising item-wise.

Characteristic	Greedy heuristic	Lagrangian heuristic	Improvement heuristic
Utility of the heuristic for the procedure	Finding of initial solution	Improvement of solution and checking of optimality	Improvement of Lagrangian result, if not already optimal
Change in SKUs per iteration	One unit of SKU removal	Multiple units of SKU removal or addition	One unit of SKU removal or addition
Relative time needed for the process	Moderate	Slow	Quick
Check for achievement of optimality	To be checked by Lagrangian procedure	Will be checked by Lagrangian procedure	Can be checked by only brute force methods

Table 7: Comparison of Heuristics

5.2.1 Comparison with Initial Greedy Heuristic

1. The initial greedy heuristic has been programmed to stop at the first feasible solution. This is a solution wherein the depots are fully filled to the maximum. The procedure stops and does not try to improve the solution. The procedure hence attains non optimal solutions especially in cases where the depots have an excess capacity with respect to the optimal solution.

2. The initial greedy heuristic removes one sku in each iteration and follows a linear path towards the solution. Hence, there exists a possibility that this heuristic stops at a local minimum rather than the global minimum.

3. The initial greedy heuristic solves the problem for each depot independently though mathematically the individual depots are not independent, i.e. the heuristic first finds the optimal order level for one of the depots and then proceeds to find the optimal order level for the second depot from the first depot's

optimal solution. However, if the process is reversed, the heuristic may reach another local optimum.

5.2.2 Comparison with Improvement Heuristic

1. The Lagrangian heuristic uses an optimality condition to check for optimality of a solution obtained. But, for the improvement heuristic there is no such optimality condition available.

2. The improvement heuristic is a local search procedure involving unit steps in any direction. So, there is a possibility that it will get stuck in a local minimum.

5.3 Example

In this example, the superiority of the Lagrangian heuristic is illustrated. The capacities of the two depots are unequal with depot 1 larger than depot 2. The demand arrival rates for items 1 and 2 at depot 1 are same but these are different at the other depot. The unit cost of item 2 is less than that of item 1. The holding costs are assumed to be the same for all the items at both the depots. Production cost for item 1 is higher than that for item 2 and so is the emergency and transshipment shipment costs. The particular values used for these parameters are provided in Tables 8 and 9. The output obtained using Initial Greedy heuristic, Lagrangian heuristic and brute force method are given in Table 10

Very clearly the initial heuristic fails to reach the global minimum. However, the Lagrangian heuristic reaches the optimal solution since it solves for solution for all the depot jointly, unlike the initial heuristic.

Parameter	Values
Capacity of depot 1 (m_1)	12
Capacity of depot 2 (m_2)	5
Arrival rate λ_{11}	7
Arrival rate λ_{12}	7
Arrival rate λ_{21}	6.5
Arrival rate λ_{22}	5

Table 8: Input parameters for the example to show the superiority of Lagrangian heuristic

Parameter	Units	Item1	Item2
Unit holding cost per period - depot 1	\$/SKU/time	1	1
Unit holding cost per period - depot 2	\$/SKU/time	1	1
Production cost of items	\$/SKU	5	2
Emergency cost	\$/SKU	20	10
Transshipment cost from depot 1 to 2	\$/SKU	10	4
Transshipment cost from depot 2 to 1	S/SKU	10	5

Table 9: Cost parameters for the example to show the superiority of Lagrangian heuristic

As the example shows, the optimal solution for depot 1 considered alone is $(s_{11}, s_{12}) = (8, 4)$. However, $(s_{11}, s_{12}) = (7, 5)$ is the optimal solution. Though the initial greedy heuristic reaches the point $(s_{11}, s_{12}) = (7, 5)$ in its route to $(s_{11}, s_{12}) = (8, 4)$, the subsequent iterations improve the solution for depot 1. However, this improvement in solution is essentially not an improvement for the overall solution. The initial greedy heuristic first optimizes the solution for depot 1 i.e., $(8, 4)$ and then tries to optimize the overall system by improving the solution for depot 2. Hence, for the above example the initial heuristic does not converge to the optimal solution.

Method	s_{11}	s_{12}	s_{21}	s_{22}
Initial Greedy Heuristics	8	4	5	0
Lagrangian Heuristics	7	5	5	0
Brute Force	7	5	5	0

Table 10: Output for example on superiority of Lagrangian heuristic

CHAPTER VI

NUMERICAL ANALYSIS

6.1 Introduction

This chapter explains the numerical experiment we have conducted to study the system reflecting real life scenarios.Sensitivity analysis is also performed. The primary focus has been convergence to optimality of the heuristics and the sensitivity analysis for changes in costs. We have selected a range of values for the parameters that are realistic for the e-fulfillment industry. Further we have pegged the arrival rates to the capacities of the depots so as to ensure that an unreasonable arrival rate is not generated for a specific capacity. Finally we have studied the extendability of the two-item problem to multi-item problem and have provided examples for the same.

6.2 Parameters for the Experiments

We have considered the book e-tailing industry as a source for the parameters used in the experiments. The two-item two-warehouse example is considered. A total of 288 experiments was performed with various combinations of parameters as listed in Table 11.

In real life situations, the inventory allocated to the warehouse and the capacity of the depot are inter-related. A serious mismatch of the capacity of the warehouse and inventory intended for the warehouse provides us with a capacity planning problem rather than an inventory allocation one. Hence the arrival rates are derived from the capacities of the warehouse through solving the following formulas with a safety factor derived using the z value of standard normal distribution.

Name of Parameter	Units	Instances	Values
Capacity of warehouse 1 (m_1)	SKUs	3	20,40,60
Capacity of warehouse 2 (m_2)	SKUs	1	60
Z value		2	2.57583,1.95996
p_1 - % of item 1 for warehouse 1	%	2	25%,50%
p_2 - % of item 1 for warehouse 2	%	2	25%,50%
Holding cost interest rate (h)	$/SKU-yr	3	0.01,0.03,0.05
Production cost of item 1 (c_1)	$/SKU	1	20
Production cost of item 2 (c_2)	$/SKU	1	20
Emergency Cost	$/SKU	2	50,60
Transshipment Cost	$/SKU	2	30,40
Total number of settings		288	

Table 11: Input parameters for the experiments

Solving simultaneously the equations (11 and 12) for r_1, r_2, we get the total arrival allocations for each depot given z, m_1, m_2:

$$r_1 + z\sqrt{r_1} - m_1 = 0 \tag{11}$$

$$r_2 + z\sqrt{r_2} - m_2 = 0 \tag{12}$$

To find the arrival rates for each of the warehouses, we solve simultaneously the equations (13, 14, 15 and 16).

$$\lambda_{11} = p_1 r_1 \tag{13}$$

$$\lambda_{12} = (1 - p_1) r_1 \tag{14}$$

$$\lambda_{21} = p_2 r_2 \tag{15}$$

$$\lambda_{22} = (1 - p_2) r_2 \tag{16}$$

The rationale for the usage of the above formulas are to peg the arrival rates to the capacity through the formula capacity = arrival rate + safety stock, the safety stock is given by $z\sqrt{r_1}, z\sqrt{r_2}$ respectively. The costs are chosen to represent a SKU of an online vendor. A major component of the holding cost is the interest payable for the SKU. Hence the holding cost per SKU is dependent on the interest rate. Three specific interest rates are considered 0.01, 0.03, 0.05. The holding cost per inventory

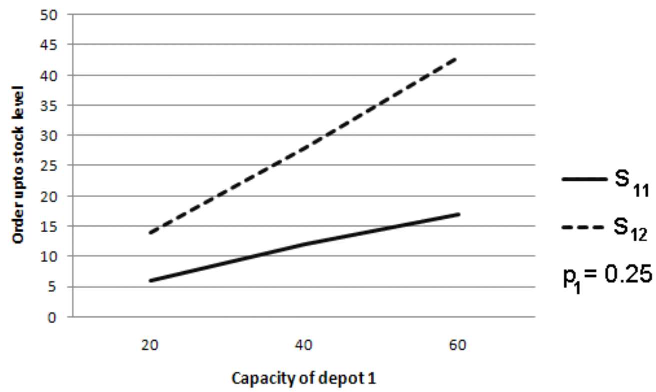

Figure 5: Plot of order level with capacity increase

per year is calculated by the product of the interest rate and procurement cost of the item divided by the number of weeks (assume 50 weeks per year). The analysis of the 288 experiments are given below.

6.3 Analysis of the Solutions

The cases listed above were solved by the initial greedy heuristic itself and optimality of the solution obtained was verified using the Lagrangian method. We recall here the sporadic cases outside of this experiment for which the initial greedy heuristic failed to provide the optimal solution and the Lagrangian heuristic was able to reach the optimal solution. Please refer to Chapter 5.

6.4 Sensitivity Analysis with respect to capacities

In our problem, the optimal solution is obtained when either one or all of Λ is zero or the warehouse is fully filled to the capacity. If any of the Λs is zero then the constraint is irrelevant. In all our experiments, the optimal solutions filled the warehouses to their capacities. When the capacity of the warehouses increased, the inventory levels also increased to occupy the larger capacity.

In the Figure 5, as capacity of depot changes from 20 to 60, the optimal order

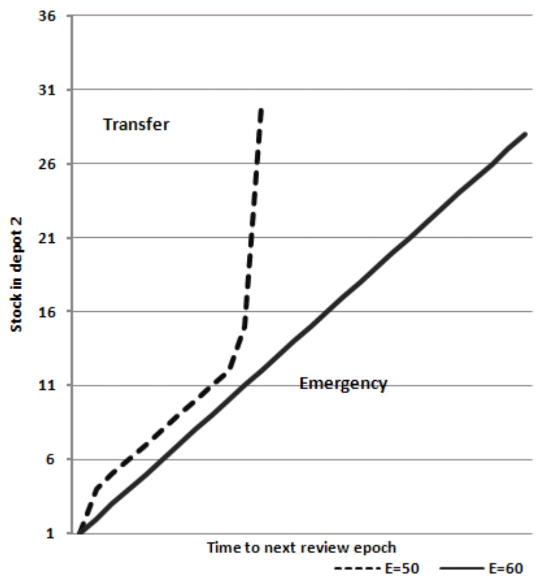

Figure 6: The sensitivity of decision on stockout with changes in emergency cost

level for the respective item in the depot increases proportionally.

6.5 Sensitivity Analysis with respect to costs

We have considered for the experiment two values each of emergency and transshipment costs and 3 values of holding costs. The solution was not sensitive to these change in these costs. Since the solution fills the inventories to the capacity, the extra cost does not alter the solution. However, 'τ_{ij}^{k}' the time instant which determines the choice between transshipment and emergency in case of stock outs changes. For increase in holding cost, the values of the time limit stay the same. Increase in emergency cost increases the time limit for which transshipment is chosen.

In the Figure **??** the lines denote the region in which the decision on stockout changes from transshipment to emergency shipment. The decision depends on two variables - the time till the next review epoch and the inventory present in the other depot. As we see in that figure, as the emergency cost increases, the region in which emergency shipments are preferred to transshipment shrinks.

CHAPTER VII

TWO-DEPOT MULTI-ITEM MODEL

7.1 Introduction

In this chapter, we demonstrate the usability of our solution procedure to multi-item two-depot models. To our knowledge, joint optimization for multi-items two-depot periodic review inventory has not been considered in the literature. We provide below a numerical example for a three-item two-depot case to illustrate this, as extension to more than three items is straightforward. This is facilitated due to the structure of the Lagrangian relaxation problem as explained in the previous chapters. For ease of reference, we recall the Lagrangian relaxation problem for two item case below:

$$\min \sum_{j=1}^{2} V_j\left(s_{1j}, s_{2j}\right) + \Lambda_1 \left(\sum_{j=1}^{2} s_{1j} - m_1\right) + \Lambda_2 \left(\sum_{j=1}^{2} s_{2j} - m_2\right) \tag{17}$$

$$= min \sum_{j=1}^{2} F\left(s_{1j}, s_{2j}\right) + \Lambda_1 m_1 + \Lambda_2 m_2 \tag{18}$$

In the above, it is easy to see that the objective function is separable and this makes it easy to extend the problem to n items. In the following section we formulate the primal problem and then the Lagrangian relaxation problem for the multi-item case.

7.2 Formulation of multi-item problem

We consider a n-item periodic review inventory system served by a central source with the two warehouses acting as storage and selling locations. Rest of the assumptions is similar to the two item case explained in the previous chapters. The problem formulation for multiple item inventory problem for n-item and two-depot case is:

Problem 2 (P-M0) *Let the number of items be n and let the number of depots be 2. Using the notation of the previous chapters, the n-item 2-depot problem optimization problem is:*

$$\min \sum_{j=1}^{n} V_j(s_{1j}, s_{2j}) \tag{19}$$

$$\textit{such that} \tag{20}$$

$$\sum_{j=1}^{n} s_{1j} \leq m_1 \tag{21}$$

$$\sum_{j=1}^{n} s_{2j} \leq m_2 \tag{22}$$

Similar to the two-item case, we introduce Lagrangian Multipliers and convert our problem into a Lagrangian relaxation problem as follows:

Problem 3 (P-M1)

$$\min \sum_{j=1}^{n} V_j(s_{1j}, s_{2j}) + \Lambda_1 \left(\sum_{j=1}^{n} s_{1j} - m_1 \right) + \Lambda_2 \left(\sum_{j=1}^{n} s_{2j} - m_2 \right) \tag{23}$$

As in the two-item case, the formulation assumes that the SKUs for all items are in terms of pallets and hence occupy the same space in the depot. Further, the ordering for stock till the next review epoch, the return of surplus for the present review period and arrival of the regular stock at the same interval of time for all items.

7.3 Example of three item inventory problem

With the problems **P-M0**, **P-M1** as formulated, a similar procedure of Initial greedy heuristic, Lagrangian heuristic and Improvement heuristic were applied for a three item example. The values of the parameters in the three item example is listed in Tables 12 and 13.

These examples were developed to explain the extendability of the 2 item case into a 3 item case. The results are displayed in Table 14. As we see, optimality was

Parameter	Units	Values
Capacity of warehouse 1 (m_1)	SKUs	30
Capacity of warehouse 2 (m_2)	SKUs	40
Z value		2.57583
Arrival rate %s for item 1	%	50%,30%,20%
Arrival rate %s for item 2	%	20%,30%,50%
Holding cost interest rate (h)	$/SKU-yr	0.01

Table 12: Input parameters for the multi-item example

Parameter	Units	Item1	Item2	Item3
Production cost of items	$/SKU	20	20	25
Emergency Cost	$/SKU	30	50	60
Transshipment Cost from depot 1 to 2	$/SKU	15	25	30
Transshipment Cost from depot 2 to 1	$/SKU	20	25	40

Table 13: Cost parameters for the multi-item example

achieved in this case too. Further, it proves that the procedure can be extended to n item cases. As noted earlier, the optimal solution fills the warehouses to their capacities.

Method	s_{11}	s_{12}	s_{13}	s_{21}	s_{22}	s_{23}
Initial Greedy Heuristic	14	9	7	8	13	19
Lagrangian Heuristic	14	9	7	8	13	19
Brute Force	14	9	7	8	13	19

Table 14: Output for multi-item example

CHAPTER VIII

CONCLUSION

In this thesis, we considered a two-depot two-item periodic inventory problem with regular, transshipment and emergency shipments. We developed the procedure for solving the problem through the usage of greedy and Lagrangian relaxation based heuristics. The procedure to verify whether the solution is optimal. Further, the procedure provides insights into the decision to be taken on facing a stockout - whether to call for emergency shipments or transshipment. When a warehouse is requested for transshipment, the decision depends on two factors - the inventory level in the warehouse and the time till the next review epoch. The higher the inventory present, higher are the chances for transshipment.

To summarise, the procedure obtains a solution for a two-depot two-item periodic inventory problem, verifies its optimality and decides timeframes for either emergency or transshipment on stock-out.

The experiments that were performed provide insights into the quality of solutions. The main insights are:

1. The initial greedy procedure is itself quite good based on the number of times the greedy heuristic has converged to the optimal solution. The solution obtained in our experiment converged in the initial heuristic itself.

2. The initial heuristic provides a feasible solution for the Lagrangian relaxation procedure. Lagrangian procedure converges either to an optimal solution or provides us with the bounds for the solution obtained.

3. Increase in emergency cost increases the time limit for which transshipment is

chosen.

4. Further, the decision to use emergency shipment is determined by the inventory in stock in the other depot.

We note that the Lagrangian relaxation approach results in a separable optimization problem which actually can be exploited to extend the problem to multi-item two-depot cases. This has also been illustrated with a two-depot three-item periodic inventory model. Thus, this thesis has demonstrated the ease with which these problems can be solved to optimality.

The work can be extended in the following directions. The study of the procedure with respect to multi-item formulation has been already discussed. The procedure can be studied to extend the number of depots as well which is more useful from a business standpoint. Studies can also be performed to reduce split delivery scenarios using the above procedure. Split delivery scenario is a scenario in which an order is handled by two or more depots due to a stockout in depots. The customer receives more than one consignment and maybe on various dates. To minimise customer dissatisfaction due to split delivery, the part shipments of that single order can be consolidated at a single place resulting in a higher time window needed for delivery. Hence, there is a need to balance the tradeoff between customer dissatisfaction and higher cost. This is a worthy extension to consider in the future for the e-fulfillment business case.

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