Dynamic Pricing with Duration Consideration in Rental Business

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Abstract

We consider the basic problem of a service firm that offers a product for rental to its customers for a fixed rental duration. Many firms prefer a predetermined rental duration to simplify the rental/return process and offer a fixed duration, such as two days, to appeal to most of its potential clients. As customer demand is normally stochastic and dependent on the rental rate, the firm has to determine not only its initial stock level but also the rental rate for its product to maximize its total profit. We present a two-stage discrete-time model to examine the rental problem. In stage one, the total time horizon is divided into N days and returned units are added to the stock at the beginning of each day. In stage two, the first day is divided into K time periods, each period is a small time interval Δt. At each time period, we dynamically adjust the rental rate according to the stock level inventory on hand and expected rental demand. We obtain the optimal pricing policy for this problem and show the existence and uniqueness of the optimal price policy. We demonstrate the expected optimal revenue is concave in the inventory level through numerical analysis. Meanwhile, we analyze the effects of the potential market size and price sensitive index by numerical studies. We find the optimal solution of the deterministic demand and show the optimal revenue is concave in the inventory on hand. Finally we show the deterministic case is the upper bound of stochastic problem and the periodical fixed price policy is asymptotical optimal as the inventory and the customer arrival go infinity.

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1 Introduction

In recent time, the accelerating growth of the world economy has stimulated an explosive development in rental industry. Some news report that the market size of the rental business reaches hundreds of billions these days. A person or a company can now rent a wide range of things- from personal to industry products, equipment and facilities. Aircraft, oil rigs, ships, apartments, lockers, safe deposit boxes, furniture, books, and videos are now offered for rent. As diverse as the items available for rentals, different rental practices exist in the rental service businesses. Some firms, for example, offer customers the flexibility to choose different rental durations that best suit their individual needs. Many others prefer offering their customers a pre-specified rental duration to better manage the rental/return processes and the utilization of their rental items. Products such as books, videos, weekend chalets are often offered for rental for a pre-specified duration.

Our generic problem comes from the pre-specified duration rental business. In the pre-specified duration rental business, retailer offer an arrive customer a pre-specified duration to use a product or service and the customer pay for it and return the product before due day. In this paper, we consider a service firm that offers a single or multiple units of a product for rental to its customers. The product is offered for rental to potential customers over a rental horizon of N days, the life-cycle of the product. At the beginning of the rental horizon, the firm must decide on the number of units that it wishes to offer to its customers for rental. Each unit rented to a customer is assumed to be leased out for a fixed rental duration, pre-set by the firm to meet the need of all its customers. The customer arrival rate is assumed to be stochastic, non-stationary and dependent on the rental rate charged by the firm; and each customer arrival is a request for one unit of the product.

If a unit is available in stock, the request of the customer is fulfilled. Otherwise, the customer turns away. On the last day of the rental duration, we assume the customer will return the rented unit promptly on the due-date. This assumption is reasonable because the rental duration is usually short and there is no incentive to return unit early before the due date. All returned units on the same day are then checked and available for rental at the beginning of the next day.

We formulate the above problem as a two-stage dynamic pricing model. The objective of the firm is to maximize its total profits by determining the optimal number of units available at the start of the rental horizon and adjusting the rental rates dynamically over the entire rental horizon time. Figure 1 illustrates the rental problem described above.

Our model provides the following insights and results to the rental service problem examined in our paper:

1. We find the optimal price policy of the rental problem. We also show the existence and uniqueness of this price policy. Base on this optimal policy, we use numerical method to demonstrate the optimal expected revenue is monotone and concave in the inventory level. We also study the effects of potential market size and price sensitive index by numerical analysis.

2. We compare the performance of the optimal price policy under different combination of analysis periods K and rental duration r. The result suggests the time properties of the optimal expected revenue broken because the existence of return process.

3. We provide optimal policy price for deterministic demand and compare the differ-

ences between single period and multi periods problem. We also present the optimal solution of the deterministic demand. The optimal revenue of deterministic case is concave in the inventory on hand.

4. The deterministic demand provides an upper bound of the stochastic problem, we prove the periodical fixed price policy is asymptotically optimal as the demand and the inventory go infinity. We also compare the performance of the optimal price policy and the deterministic demand solution.

The rest of the paper is organized as follows. In Section 2, we review the existing literature related to our paper. In Section 3, we present the problem and our model. We then derive the optimal pricing policy and give the optimal solution through numerical analysis in Section 4, we also analyze the effects of the coefficient in this part. Based on a deterministic version of our model, we derive the optimal policy and provide optimal closed form solution in Section 5, we also present the periodical fixed price policy and test the optimal policy's performance in this part. We provide a summary of my thesis and talk about the future work in the last section.

2 Literature Review

Revenue management is a very active research area in operations management these days. The study of revenue management begins in airlines industry which dates back to 1972. Littelewood (1972) [15] studies a stochastic two-fare, single-leg problem. This paper describes the early work on applying mathematical models to the development of revenue management in the airline industry. After nearly 40 years' progress, researchers and practitioners have done enormous of works on different areas in revenue management, such as the economics implications of revenue management; the pricing, capacity control, overbooking and forecasting problems of revenue management; how to implement revenue management in different industries and how to evaluate the impact of revenue management.

Gallego and van Ryzin (1994) [10] present a very important and fundamental paper on the dynamic pricing problems of revenue management. They focus on the problem how to operate a given stock of one product with stochastic demand by a deadline. They show the optimal expected revenue is concave in the inventory number and left of time, prove the deterministic case is the up-bound of the stochastic one. Base on this result they give a fixed-price heuristic which is asymptotically optimal when the volume of expected sales is large. They also extend their results to compound poisson demand, etc. After that, Gallego and van Ryzin (1997) [11] analysis a multiproduct dynamic pricing problem and its applications to the network yield management. In this paper, they first formulate a m

kinds of resources and n kinds of products stochastic model. Because the difficulty of model, instead of giving any theoretic properties about the stochastic models, they show the deterministic case is the up-bound for stochastic case, and give two heuristics base on this property. The result of the numerical examples provide some fundamental insights into the performance of revenue management system.

Bitran and Caldentey (2003) [3] present a comprehensive overview of the pricing models for revenue management. They discuss the research works already done in both single product problem and multiproduct problem in this paper. They also provide a unified structure for the pricing problem in revenue management. As they mention, except the work by Gallego and van Ryzin (1997) [11], most of papers which analysis stochastic multiproduct problems focus on the static price model.

Maglaras and Meissner (2006) [16] study a single resource but multiproduct stochastic problem. Not as other papers in stochastic multiproduct problems, they introduce a dynamic pricing model and which could give a unified analysis of both pricing and capacity control problem. In this unified model, they show the optimal expected revenue is concave in both the inventory level and the left periods of time. For the deterministic case, they use fluid model to get the optimal condition and asymptotically optimal heuristics. The difference between our paper and theirs are our paper consider a multiple days rental problem, the return process makes our paper become a multiple-resources problem.

The book by Talluri and van Ryzin (2005) [20] gives an comprehensive review of revenue management to the people who want to learn more about this research area. Another reviews papers include Elmaghraby and Keskinocak (2003) [8] and Chiang, Chen and Xu (2007)[6].

A specifical stream of research work which related to ours focus on the rental business. Carroll and Grimes (1995) [5] and Geraghty and Johnson (1997) [13] tell us how the revenue management change the car rental industries from empirical points of view. Anderson and Blair (2004) [1] introduce the opportunity costs to measure the revenue management performance in a car rental companies.

Utilizing the panel data collected at 6137 video rental stores in the US between 1998 and 2000, Mortimer (2004) [17] analyzes the effects of vertical contracts in video rental industry. She compares the different pricing contracts, pricing contracts and revenue-sharing contracts, etc., across different stores for the same title as well as across different titles within the same store. Her analysis shows that the revenue sharing scheme increases profits for both upstream and downstream firms and consumer surplus.

Bayiz and Tang (2004) [2] develop an integrated planning system for a dosimetry service company, which is the first to integrate various operations-research techniques for managing refurbished products with both uncertain demand and return, and supplier constraints.

Gans and Savin (2007) [12] analyze a pricing and capacity rationing rental problem with uncertain durations. They use a transformation to simplify the initial high dimensions model to a queueing model. In this queueing model, the arrival rate and rental process are not affected by rental price. They show the optimal expect revenue is concave in the number of items on hand in this paper. They also show the myopic policy is asymptotically optima as both the offered load and system capacity become large in which. The differences between their paper and ours first is we choose our model is intensity control model, second is our model include price as a decision variable.

Tang and Deo (2008) [21] consider a rental problem with price and duration. They use the multi-period pricing newsvendor model to analysis this problem. They prove the optimal expected revenue is concave in T when the market potential is high. They also prove the optimal rental price is increasing and concave in the rental duration when the return process is smooth. After analyzing the original model, they study the revenue sharing and retailer competition cases in rental business. Not as their paper usees only one price cover the whole time horizon, my theses usees dynamic pricing to analysis the rental problem.

Ding, Lim and Yang (2008) [7] is a recently paper working on the multiple-capacity dynamic pricing problem of revenue management. They study a multiple-days stay problem in hotel business, and which belongs to the multiple-resources and multiple-components problems. They show the expected net benefit of receiving a reservation is concave in the rental rates given the inventory level and the single-day approximation is asymptotical optimal when the left rooms are ample. They also give the heuristics based on optimal policy and single-day approximation respectively. The difference of my thesis is we study a two stages problem, and the second stage is a multiple-resources single-component problem.

3

Model

After we review the related work of my thesis, we present the notations we use:

$\lambda_{k,t}$	The Poisson demand rate in k day and time period t
$p_{k,t}$	The rental price in k day and time period t
$r(\cdot)$	The revenue rate function
$\Pi_{k,t}$	The optimal expect revenue from period t of k day
Δ_n	The marginal optimal expected revenue in inventory level
Δ_t	The marginal optimal expected revenue in time period
Δ_{n_i}	The marginal optimal expected revenue in rental information
Δ_k	The marginal optimal expected revenue in different day
$\Pi^D_{k,t}$	The optimal expect revenue from period t of k day in deterministic case

The meaning of these notations will be well explained when we use them.

3.1 Model Description

After we introduce the notation, we begin describe our model: We consider a videos rental company which operates N items of video over a period of K days, and each day we can divided to T time periods. The rental duration of the videos is r days. To simplify the problem, we assume the customers follow Poisson Arrival. Because the demand is price sensitive, we define the arrival rate is $\lambda(p)$. To simplify the problem, we have a basic assumption about the rental items:

Assumption 1. *The items rent out could just be returned after r days, and the items returned can't be rent out in the same day.*

This assumption is reasonable, because when the rental duration is short, the customer have non incentive to return the items before due day. Meanwhile, the company need some time to operate the inventory, such as clean or make sure it's not broken.

We also assume the revenue rate $r(\lambda) \doteq p\lambda(p) = p(\lambda)\lambda$. Moreover, $r(\lambda)$ is a regular function, from the common definition in revenue management, $r(\lambda)$ satisfies $\lim_{\lambda\to 0} = 0$, is a continuous, bounded and concave, and has a bounded least maximizer defined by $\lambda^* = \min\{\lambda : r(\lambda) = \max_{\lambda\geq 0} r(\lambda)\}$. Then we can build up our formulation based on these assumptions.

3.2 Formulation

To cover these specialities of the problem, we use two stages discrete time dynamic pricing model. In the first stage, we operate the inventory, which means at the beginning of each day, we add the item returned in last day to the inventory level. In the second stage, we dynamic price and rent the video to arrive customer. We divide one day to T periods, and the length of each period is δt. When δt is small, from the definition of Poisson arrival, at most one customer could come and rent a item.

In the second stage of the problem, we formulate the problem in one day as follows: In day k, time period t, the firm has a stock and rental information $(n, , n_1, \ldots, n_{r+1})$, which satisfies $\sum_{i=1}^{r+1} n_i + n = N$. The company has t time periods to rental these n items out, or to leave it for next day. in this stage, the firm control the intensity of Poisson demand $\lambda_{k,t} = \lambda(p_{k,t})$ by using a non-anticipating pricing policy $p_{k,t}$ at time period t. We explain the meaning of rental information $(n, n_1, \ldots, n_r, n_{r+1})$. n_i means the items will be returned

to the system $i-th$ day later and n means the inventory we can rent out in this t time periods. We already assume the intensity $\lambda(\cdot)$ is a regular demand function. Use $\Pi_{k,t}$ presents the optimal expected revenue get from period t of day k. Base our setting, we use back-forward dynamic programming to formulate our model:
When $k = K+1$, because the rental business is already close, so

$$\Pi_{K+1,1}(n, n_1, n_2 \ldots, n_r, n_{r+1}) = 0$$

When $t = T+1$, the second stage is end, we go into the first stage and operate the inventory. We add the return items of last day to the inventory on hand at the time period.

$$\Pi_{k,T+1}(n, n_1, \ldots, n_r, n_{r+1}) = \Pi_{k+1,1}(n+n_1, n_2, \ldots, n_{r+1}, 0)$$

We can get the expression of our model by considering what happens over a small interval of time δt. Since by selecting the intensity λ (i.e., pricing at $p(\lambda)$) we rent out one item over the next δt with probability $\lambda\delta t$ and no items rent out with the probability approximately $1-\lambda\delta t$, by the Principle of Optimality. For the last period of last day

$$\begin{aligned}
\Pi_{K,T}(n, n_1, \ldots, n_r, n_{r+1}) &= \sup_{\lambda}[\lambda\delta t(p(\lambda) + \Pi_{K,T+1}(n-1, n_1, \ldots, n_r, n_{r+1}+1)) \\
&\quad +(1-\lambda\delta t)\Pi_{K,T+1}(n, n_1, \ldots, n_r, n_{r+1}) + o(\delta t)] \\
&= \sup_{\lambda}[\lambda\delta t(p(\lambda)) + o(\delta t)].
\end{aligned}$$

From the last period, dynamically we have

$$\begin{aligned}
\Pi_{K,t}(n, n_1, \ldots, n_r, n_{r+1}) &= \sup_{\lambda}[\lambda\delta t(p(\lambda) + \Pi_{K,t+1}(n-1, n_1, \ldots, n_r, n_{r+1}+1)) \\
&\quad +(1-\lambda\delta t)\Pi_{K,t+1}(n, n_1, \ldots, n_r, n_{r+1}) + o(\delta t)]
\end{aligned}$$

Following the same methodology, at any day any period, we have

$$\begin{aligned}\Pi_{k,t}(n,n_1,\ldots,n_r,n_{r+1}) &= \sup_{\lambda}[\lambda\delta t(p(\lambda)+\Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)) \\ &\quad +(1-\lambda\delta t)\Pi_{k,t+1}(n,n_1,\ldots,n_r,n_{r+1})+o(\delta t)] \qquad (3.1)\end{aligned}$$

where $k=1,2,\ldots,K$ and $t=1,2,\ldots,T$.

The firm's problem is to find a pricing policy that maximizes the total expected revenue generated from the whole operating time:

$$\Pi(N)=\Pi_{1,1}(N,0,\ldots,0,0) \qquad (3.2)$$

$\Pi_{1,1}(N,0,\ldots,0,0)$ means at the first day, first period of the business, all the inventory is on hand.

We derive the optimal solution in the later part of my thesis.

4

Optimal Policy of the Problem

4.1 Optimal Price policy

We define $\Delta_t \Pi_{k,t}(n,n_1,\dots,n_r,n_{r+1}) \doteq \Pi_{k,t+1}(n,n_1,\dots,n_r,n_{r+1}) - \Pi_{k,t}(n,n_1,\dots,n_r,n_{r+1})$ and $\Delta_n \Pi_{k,t}(n,n_1,\dots,n_r,n_{r+1}) \doteq \Pi_{k,t}(n,n_1,\dots,n_r,n_{r+1}) - \Pi_{k,t}(n-1,n_1,\dots,n_r,n_{r+1})$. By using $r(\lambda) \doteq \lambda p(\lambda)$ and rearranging (1), for all $k = 1,2,\dots,K$ and $t = 1,2,\dots,T$, we obtain

$$\begin{aligned}\Pi_{k,t}(n,n_1,\dots,n_r,n_{r+1}) &= \sup_{\lambda}[r(\lambda)\delta t + \lambda \delta t \Pi_{k,t+1}(n-1,n_1,\dots,n_r,n_{r+1}+1) \\ &\quad +(1-\lambda\delta t)\Pi_{k,t+1}(n,n_1,\dots,n_r,n_{r+1})] \quad \forall n \geq 1. \qquad (4.1)\end{aligned}$$

The existence of a unique optimal solution to equation (3) is resolved by the following proposition, which is proved in the appendix:

Proposition 1. *If $\lambda(p)$ is a regular demand function, then there exists a unique solution to equation (2), further, the optimal intensities satisfies $\lambda^*_{k,t}(n,n_1,\dots,n_r,n_{r+1}) \leq \lambda^*$ for all n and k for all $1 \leq t \leq T$.*

Because the difficulties of getting the closed form solution, and proposition 1 guarantees the existence of a unique optimal solution, we can use numerical method to get the optimal solution.

4.2 Optimal numerical solution

Based on the dynamic formulation of this rental problem, we can get the optimal policy through numerical method dynamically.

We first analyze the effects of the potential market size and the price sensitive index through numerical analysis. We choose the scenario in which rental duration $r = 2$ days, analysis periods $K = 10$ days and the initial inventory is 20 items. The numerical result shows in the two pictures in figure 1: The two pictures in figure 1 try to analysis the effects through the optimal expected revenue and the optimal rental price.

The left picture of figure 1 describe the change of optimal expected revenue when we change the potential market size and price sensitive index. In this picture, the rental duration r, analysis periods K, and the initial inventory level X keep unchange, so all the differences of the optimal expected revenue come from the change of the potential market size and price sensitive index. When we look the picture along the axis of price sensitive index, we will see the curve of the optimal expected revenue going down. This means when the potential market size is fixed, as the price sensitive index increase, the optimal expected revenue will decrease. When we look the picture along the axis of potential market size, we will see the curve of the optimal expected revenue going up. This means when the price sensitive index is fixed, as the potential market size increase, the optimal expected revenue will increase. When we combine these two results, we get that the bigger potential market size and smaller price sensitive index will provide greater optimal expected revenue.

The right picture of figure 1 describe the change of the optimal rental price when we change the potential market size and price sensitive index. In this picture, the rental duration r, analysis periods K, and the initial inventory level X keep unchange, so all the differences of the inventory utilization rate come from the change of the potential market size and price sensitive index. When we look the picture along the axis of price sensitive index, we will see the curve of the optimal rental price going up. This means when the potential market size is fixed, as the price sensitive index increase, the optimal rental price will increase. When we look the picture along the axis of potential market size, we will see the curve of the optimal rental price going down. This means when the price sensitive index is fixed, as the potential market size increase, the inventory optimal rental price will decrease. When we combine these two results, we get that the smaller potential market size and bigger price sensitive index will provide greater optimal rental price.

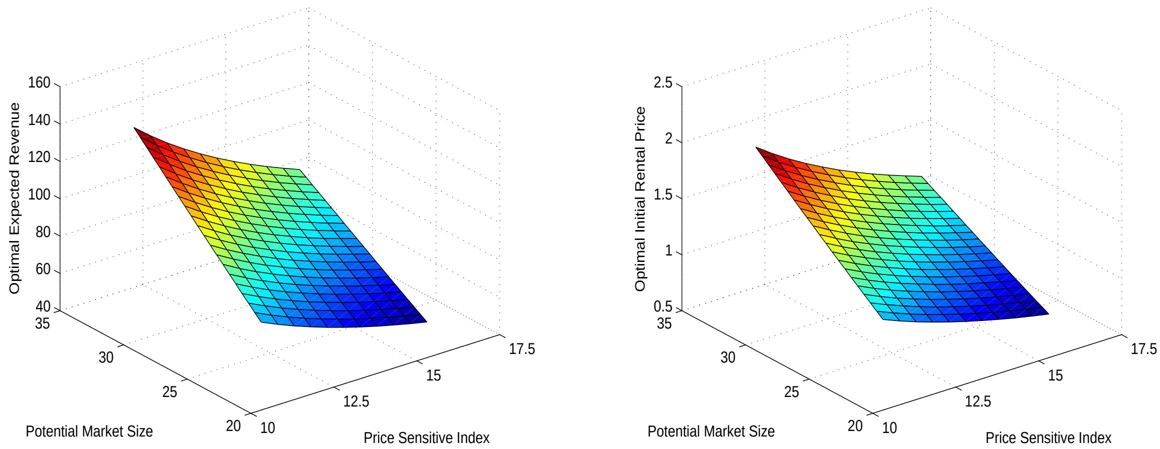

Figure 4.1: The Optimal Expected Revenue and Optimal Rental Price with different potential market size and price sensitive index

After analysis the effect of potential of market size and price sensitive index, we come to study the effect of initial inventory level. The study base on the scenario in which the potential market $a = 20$ items, price sensitive index $b = 10$ items/dollar, rental duration $r = 3$ days and the analysis period is $K = 10$ days. Figure 2 the change of the optimal expected revenue and initial optimal rental price when we change the initial inventory level. The left picture of Figure 2 shows the optimal expected revenue increase as the inventory level and when the inventory hit a bound, the optimal revenue maximized. The picture also prompt the optimal expected revenue is concave in inventory level, this proposition we still haven't prove it, we will mention it in the later remarks. The right picture of Figure 2 shows the initial optimal rental price decrease as the inventory increase and when the inventory hit a bound, the initial rental price minimized. The picture also prompt the initial optimal rental price is convex in inventory level. This property is relate to the optimal expected revenue is concave in the inventory level. Not only the initial optimal price have this property, all the optimal rental price following it.

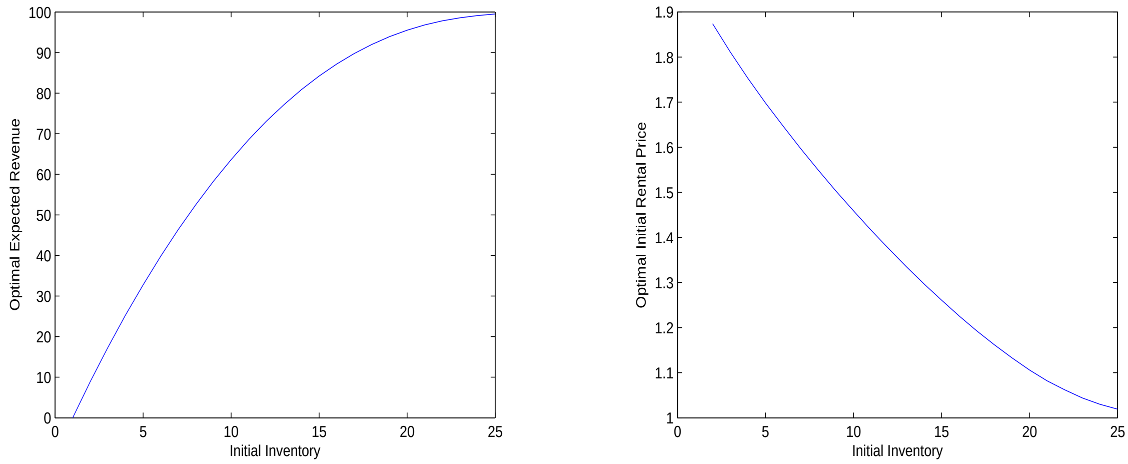

Figure 4.2: The Optimal Expected Revenue and Optimal Rental Price with different initial inventory

Based on the observation of figure 2, we give the following remark about the structure property of the optimal expected revenue about the inventory level:

Remark 1. *The optimal expected revenue* $\Pi_{k,t}(n, n_1, \ldots, n_r, n_{r+1})$ *is strictly increasing in n, and the marginal optimal expected revenue* $\Delta_n \Pi_{k,t}(n+1, n_1, \ldots, n_r, n_{r+1})$*, is decreasing in n and t:*

$$\Delta_n \Pi_{k,t}(n+1, n_1, \ldots, n_r, n_{r+1}) \leq \Delta_n \Pi_{k,t}(n, n_1, \ldots, n_r, n_{r+1})$$

And the optimal expected revenue $\Pi(N) = \Pi(N, 0, 0, \ldots, 0)$ *is concave in N.*

This is a very important policy, because we can get the optimal inventory level through these property. The return process makes the proof of this property much more difficult than the proof of the concavity of the optimal expected revenue without return.

After analysis the properties of inventory in Rental Problem, we continue our work in the effect of time period and rental duration.

4.3 The time properties of the optimal expected revenue

To analyze the effects of analysis periods and rental duration, we use numerical simulation. We focus on the scenario in which the potential market size is 20 items, the price sensitive index is 10 items/dollar and the purchase cost for per is 3 dollar/item: the average rental price and average served customer based on the 1000 times simulation. When $r = 1$, this problem become the normally revenue management problem. Everyday the retailer face a single day dynamic pricing problem. This problem is solved by Gallego and van Ryzin (1994).

We use four targets to judge the performance of optimal policy under different combination of K and r. We choose analysis periods $K = 3, 6, 9, 12, 15, 18$ days and rental duration $r = 1, 2, 3$ days, the potential market size $a = 20$ items and the price sensitive index is $b = 10$ items/dollar.

When rental duration $r = 3$, analysis periods $K = 15$ and $K = 18$, because the computation complexity, the numerical result beyond the ability of Matlab. We don't have the numerical results for these two case. If the analysis periods is $K = 3$, because the revenue of the rental business is smaller than the purchase cost, so there is no numerical results for these case.

The table 'Optimal Expected Revenue and Average Rental Price with different K and r' describe the influence of the analysis period K and the rental duration r to the optimal expected revenue and average rental price. We first analysis the change of optimal expected revenue Π^* in this table. The first phenomenon we find is under the same rental duration, the bigger analysis periods with big optimal expected revenue, but the increase number is fluctuated. Under the same analysis period, the short rental duration has high optimal expected revenue. This is easy to understand, because short rental duration means a quick cycling of the inventory. The gap of the optimal expected revenue between two rental duration change fluctuated as the analysis periods increase. Another phenomenon of the optimal expected revenue is: when the analysis period K become divided by r, the increase of the optimal expected revenue is bigger. Then we focus the average rental price. With different combination of analysis periods K and rental duration r: the first phenomenon we find is long rental duration with higher average rental price under the same analysis periods. The gap of the average rental price between two rental duration under the same analysis periods decrease as the analysis periods increase. Another phenomenon is the average rental price is decrease as the analysis periods increase with the same rental duration. The gap of the average rental price between two analysis period with the same rental duration changes fluctuated as the analysis periods increase.

Optimal Expected Revenue and Average Rental Price with different K and r

	Optimal Expected Revenue			Average Rental Price		
	$r=1$	$r=2$	$r=3$	$r=1$	$r=2$	$r=3$
$K=3$	$\Pi^*=20.07$	$\Pi^*=6.41$	$\Pi^*=\emptyset$	$p^*=1.524$	$p^*=1.871$	$p^*=\emptyset$
$K=6$	$\Pi^*=52.79$	$\Pi^*=38.40$	$\Pi^*=20.29$	$p^*=1.275$	$p^*=1.562$	$p^*=1.803$
$K=9$	$\Pi^*=83.05$	$\Pi^*=72.85$	$\Pi^*=60.06$	$p^*=1.203$	$p^*=1.285$	$p^*=1.544$
$K=12$	$\Pi^*=114.48$	$\Pi^*=107.63$	$\Pi^*=97.06$	$p^*=1.140$	$p^*=1.263$	$p^*=1.395$
$K=15$	$\Pi^*=146.26$	$\Pi^*=137.84$	$\Pi^*=\emptyset$	$p^*=1.089$	$p^*=1.225$	$p^*=\emptyset$
$K=18$	$\Pi^*=175.51$	$\Pi^*=168.82$	$\Pi^*=\emptyset$	$p^*=1.090$	$p^*=1.189$	$p^*=\emptyset$

The table 'Optimal Inventory Level and Average Served Customer with different K and r' describe the influence of the analysis period K and the rental duration r to the optimal inventory level and average number of the served customer. The first objective we care of this table is the optimal inventory level N^*. When the potential market size, price sensitive index and the purchase cost is fixed, normally the bigger rental duration need high inventory level and as the analysis periods increase. This result breaks when both r and k are small. As the analysis periods increase, the gap of the optimal inventory level N^* under the same rental duration decrease. Under the same rental duration, the gap between the rental duration $r = 1$, $r = 2$ and $r = 3$ change increase. Then we study the change of number of served customer N_c with different combination of K and r. Under the same rental duration, the number of served customer increase as the analysis periods increase, but the increase of this number is fluctuated. The gap of the number of served customers between two rental duration under same analysis periods decrease as the analysis periods increase. Another phenomenon of the number of served customer is: when the analysis period K become divided by r, the increase of the number of served customer is bigger.

Optimal Inventory Level and Average Served Customer with different K and r

	Optimal Inventory Level			Average No. of Served Customer		
	$r = 1$	$r = 2$	$r = 3$	$r = 1$	$r = 2$	$r = 3$
$K = 3$	$N^* = 6$	$N^* = 3$	$N^* = \emptyset$	$N_c = 13.93$	$N_c = 3.880$	$N_c = \emptyset$
$K = 6$	$N^* = 9$	$N^* = 10$	$N^* = 7$	$N_c = 41.96$	$N_c = 25.53$	$N_c = 11.77$
$K = 9$	$N^* = 10$	$N^* = 14$	$N^* = 15$	$N_c = 68.74$	$N_c = 53.54$	$N_c = 39.88$
$K = 12$	$N^* = 11$	$N^* = 17$	$N^* = 20$	$N_c = 98.47$	$N_c = 85.03$	$N_c = 70.39$
$K = 15$	$N^* = 12$	$N^* = 18$	$N^* = \emptyset$	$N_c = 130.58$	$N_c = 111.47$	$N_c = \emptyset$
$K = 18$	$N^* = 12$	$N^* = 19$	$N^* = \emptyset$	$N_c = 157.29$	$N_c = 139.44$	$N_c = \emptyset$

Through these two table, we find that because the existence of rental duration, the second order time properties of this rental problem break. After finishing the numerical analysis of the time properties, we will study a special case of rental problem, the case with deterministic demand.

5

Deterministic Case

When the demand is deterministic, the rental problem get simplified. Before we dealing with this deterministic demand problem. We introduce some notations. First We denote the all non-anticipating pricing policies by $\mathscr{U}$ which satisfy

$$\sum_{t=1}^{T} dN_{k,t} \leq n$$

Given a price policy u_k in period k, a rental information $(n, n_1, \ldots, n_r, n_{r+1})$, we denote the expected revenue $\Pi_{k,t}^{u_k}(n, n_1, \ldots, n_r, n_{r+1})$ by

$$\begin{aligned} & \Pi_{k,t}^{u_k}(n, n_1, \ldots, n_r, n_{r+1}) \\ \doteq \quad & \mathbb{E}_{u_k}\left[\sum_{s=t}^{T} p_{k,s} dN_{k,s} + \Pi_{k+1,1}(n + n_r - \sum_{s=t}^{T} dN_s^k, N - \sum_{i=1}^{r} n_i + \sum_{s=t}^{T} dN_{k,s} - n, n_1, \ldots, n_{r-1})\right] \end{aligned}$$

where

$$\Pi_{k,t}^{u_k}(0, n_1, n_2, \ldots, n_r) = \Pi_{k+1,1}(n_r, N - \sum_{i=1}^{r} n_i, n_1, \ldots, n_{r-1}),$$

$$\Pi_{k,T+1}^{u_k}(n, n_1, \ldots, n_r, n_{r+1}) = \Pi_{k+1,1}(n + n_r, N - \sum_{i=1}^{r} n_i - n, n_1, \ldots, n_{r-1})$$

and

$$\Pi_{K+1,1}^{u_k}(n, n_1, \ldots, n_r, n_{r+1}) = 0$$

are the boundary conditions.

The firm's problem is to find a pricing policy u_k^* that maximizes the total expected revenue generated for period k, time interval t, denote $\Pi_{k,t}(n, n_1, \ldots, n_r, n_{r+1})$. Equivalently,

$$\Pi(N) \doteq \Pi_{1,1}(N, 0, 0, \ldots, 0)$$

5.1 Single day deterministic case

Before analysis the multi-periods deterministic demand problem, we watch a special case: When $K = 1$, the problem reduce to the single day deterministic problem. When the problem is a single day problem, we don't need to consider the return process. This is a normal single day deterministic revenue management problem, which is well studied by Gallego and van Ryzin:

$$\begin{aligned} \Pi^D_{K,t}(n, n_1, \ldots, n_r, n_{r+1}) &= \max_{\lambda(i)} \sum_t^T r(\lambda(i))\delta t \\ \text{subject} &\quad \text{to} \\ \max_{\lambda(i)} \sum_t^T r(\lambda(i))\delta t &\leq n \end{aligned}$$

The optimal policy for this case is fixed price policy, which means the rental price is either the run-out price or the optimal price. When the analysis periods K is greater than 1, we must include the return process to this deterministic problem.

5.2 Formulation of Deterministic Problem

Consider the following deterministic version of the problem: At the beginning of period k, the firm has a stock of x, the instantaneous deterministic demand rate is a function of the price at time interval t, and λ is a regular demand function. Given a stock $(n, n_1, \ldots, n_r, n_{r+1})$ on hand at the time interval t in period k, we denoted $\Pi^D_{k,t}(n, n_1, \ldots, n_r,$

n_{r+1}) as the maximized expected revenue of this deterministic problem:

$$\Pi^D_{k,t}(n,n_1,\ldots,n_r,n_{r+1}) = \max_{\lambda}[\lambda\delta t(p(\lambda)) + \Pi^D_{k,t+1}(n-\lambda\delta t,n_1,\ldots,n_r,n_{r+1}+\lambda\delta t)] \tag{5.1}$$

with boundary condition $\Pi^D_{k,t}(0,n_1,\ldots,n_r,n_{r+1}) = \Pi^D_{k+1,1}(n_1,n_2,\ldots,n_{r+1},0)$ and $\pi^D_{k,T+1}(n, n_1,\ldots,n_r,n_{r+1}) = \Pi^D_{k+1,1}(n+n_1,n_2,n_1,\ldots,n_{r+1},0)$.

A very useful observation for multi-periods deterministic rental problem is:

Lemma 1. *When the demand is deterministic, the fixed price policy is optimal in each period, which means in each period, using a single price is better than using dynamic price.*

Because of Lemma 1, we can simplify the problem.

$$\Pi^D_{k,1}(n,n_1,\ldots,n_r,0) \quad = \quad \Pi^D_k(n,n_1,\ldots,n_r)$$

5.3 Optimal Solution of the Deterministic Problem

Following the definition of Gallego and van Ryzin in single day problem, we define the *run-out rate*, denoted λ^0, by $\lambda^0 \doteq N/(rT\delta t)$, the *run-out price*, denoted p^0, by $p^0 \doteq p(\lambda^0)$, the *run-out-revenue* rate $r^0 \doteq p^0\lambda^0$. Recall that λ^* is the least maximizer of the revenue function $r(\lambda) = \lambda p(\lambda)$. We define $p^* \doteq p(\lambda^*)$, and $r^* \doteq p^*\lambda^*$. Extend the fixed price policy in single period deterministic problem, we have the similar properties of multi-periods deterministic problem:

Proposition 2. *When K is divided evenly by r, the optimal solution to the deterministic problem (2) is $\lambda^D \doteq \min(\lambda^*,\lambda^0)$, for all the t and k. In terms of price the optimal policy* *is* $p^D \doteq \max(p^*, p^0)$, *for all t and k. Finally, the optimal revenue is*

$$J^D_{1,1}(N) = KT\delta t \min(r^*, r^0) \tag{5.2}$$

When K isn't divided evenly by r, which means $K = nr + l$, *we divide the the days in one period to two classes, the first class includes the first l-th days in one period. the second class includes the left days. Because the first class can be rent out one more time, so the we prefer rent the items in that days. The optimal policy is the marginal revenue of first class times its rent out times = the marginal revenue of second class times its rent out times. Which means* $(n+1)\Delta r_1(x) = n\Delta r_2(x')$. *The optimal revenue is*

$$\Pi^D_{1,1}(N) = (n+1) * l * r_1(x) + n * (r-l) * r_2(x') \tag{5.3}$$

x and x' *satisfy* $lx + (r-l)x' = N$.

For all period k, the optimal expected revenue of deterministic case $\Pi^D_k(n, n_1, \ldots, n_r)$ *is concave in n.*

Proposition 2 give the optimal solution of deterministic demand problem. This solution provides a basic insight to the relationship between deterministic demand problem and stochastic problem.

5.4 Deterministic Revenue as an Upper Bound of Stochastic Case

Intuitively, one would expect that the uncertainty arrival in the stochastic problem results in lower expected revenues. Which means the following proposition:

Proposition 3. *If* $\lambda(p)$ *is a regular demand function, then for all* $0 \leq n < N - \sum_{i=1}^{r} n_i$ *and*

$1 \leq t \leq T$,

$$\Pi_{k,t}(n, n_1, \ldots, n_r, n_{r+1}) \leq \Pi^D_{k,t}(n, n_1, \ldots, n_r, n_{r+1}) \tag{5.4}$$

Proposition 3 is important because it suggests that the deterministic solution maybe provide insight to the optimal or near-optimal policies to the stochastic problem, meanwhile it can guarantee the performance of these stochastic policies. In the same time, it can be used to establish the asymptotically optimal heuristics based on the deterministic price policy.

5.5 Asymptotically Optimal Solution

From the definition of time period, and the operation time of a day is Γ, the length of each time period is $\delta t = \Gamma/T$, first T is large, in this heuristic, we try to analyze what will happen when T become smaller. When $T = 1$, the problem become to fixed price problem. When T goes infinity, the discrete time model become continuous time model.

We first introduce a price policy which we call it periodical fixed price policy: in each period, we use a fixed price which get from the deterministic problem, in the same time, we control the inventory by following way, we keep the initial inventory of each period always same to the deterministic case, to get this, we will allow the lost sale items and will keep some inventory no using until r days later.

Proposition 4. *The expected revenue of periodical fixed price policy satisfies:*

$$\frac{\Pi^*_{1,1}(N,0,\ldots,0,0)}{\Pi_{1,1}(N,0,\ldots,0,0)} \geq \frac{\Pi^{PFP}_{1,1}(N,0,\ldots,0,0)}{\Pi_{1,1}(N,0,\ldots,0,0)} \geq 1 - \frac{1}{2\sqrt{\min\{N/r, \lambda^* \cdot (T-t+1)\delta t\}}}$$

or

$$\frac{\Pi^{PFP}_{1,1}(N,0,\ldots,0,0)}{\Pi_{1,1}(N,0,\ldots,0,0)} \geq 1 - \frac{1}{2\sqrt{\min\{x', \lambda^* \cdot (T\delta t)\}}}$$

where x satisfies $lx + (r-l)x' = N$ and $(n+1)\Delta r_1(x) = n\Delta r_2(x')$.

Because optimal policy is better than the periodical price policy. So when the the inventory is high and the number of rent out items is big, The performance of the optimal policy should be close to the deterministic demand problem.

We test the gap between the optimal policy and the deterministic policy under the scenario in which the price sensitive index is 10 items/dollar, the analysis period is 12 periods, the purchase cost of each item is 3 dollar. We compare the performance under different potential market size.

The following two tables include the simulation performance of the optimal stochastic policy and the deterministic demand case. We will compare the performance of the optimal stochastic policy with the deterministic demand problem. The table 'Optimal Inventory Level and Expected Revenue with different K and r' focus on the optimal inventory level and optimal expected revenue. The numerical result support the deterministic demand problem is the upper bound of the stochastic problem, as the potential market size increase, the optimal expected revenue of optimal policy gets closer to the deterministic upper bound. From the table, we find the revenue of the optimal stochastic policy can get nearly 98% of the deterministic demand revenue. The inventory level of the optimal stochastic policy is higher than the deterministic demand, because the need of filling uncertain demand.

Optimal Inventory Level and Expected Revenue with different K and r

	Optimal Inventory Level			Optimal Expected Revenue		
	Optimal	Deterministic	Error	Optimal	Deterministic	Error
$a = 20$	$N^* = 17$	$N_D = 16$	6.25%	$\Pi^* = 107.63$	$\Pi_D = 115.20$	6.57%
$a = 22$	$N^* = 19$	$N_D = 18$	5.55%	$\Pi^* = 132.41$	$\Pi_D = 140.40$	5.69%
$a = 24$	$N^* = 21$	$N_D = 20$	5.00%	$\Pi^* = 159.70$	$\Pi_D = 168.00$	4.94%
$a = 26$	$N^* = 23$	$N_D = 22$	4.54%	$\Pi^* = 189.53$	$\Pi_D = 198.00$	4.28%
$a = 28$	$N^* = 26$	$N_D = 24$	8.33%	$\Pi^* = 224.91$	$\Pi_D = 230.40$	2.38%
$a = 30$	$N^* = 28$	$N_D = 26$	7.69%	$\Pi^* = 259.88$	$\Pi_D = 265.20$	2.01%

The table 'Average Served Customer and Rental Price with different K and r' focus on the performance of the average number of the served customer and the average rental price. The optimal rental price closes to the deterministic rental price, as the potential market size increase. Meanwhile, the number of served customer close to the deterministic one.

Average Served Customer and Rental Price with different K and r

	Average No. of Served Customers			Average Rental Price		
	Optimal	Deterministic	Error	Optimal	Deterministic	Error
$a=20$	$N_c = 84.98$	$N_c = 96$	11.48%	$p^* = 1.263$	$p_D = 1.20$	5.25%
$a=22$	$N_c = 96.28$	$N_c = 108$	10.85%	$p^* = 1.365$	$p_D = 1.30$	5.00%
$a=24$	$N_c = 107.65$	$N_c = 120$	10.29%	$p^* = 1.467$	$p_D = 1.40$	4.78%
$a=26$	$N_c = 119.12$	$N_c = 132$	9.76%	$p^* = 1.569$	$p_D = 1.50$	4.60%
$a=28$	$N_c = 135.36$	$N_c = 144$	6.00%	$p^* = 1.630$	$p_D = 1.60$	1.88%
$a=30$	$N_c = 146.99$	$N_c = 156$	5.78%	$p^* = 1.726$	$p_D = 1.70$	1.53%

Totally speak, the optimal rental policy performance asymptotically close the deterministic demand solution.

6 Conclusion

I conclude my thesis in this part:

Nowadays, rental business has hundreds of billions market. The lack of the theoretic research work on this area is the original motivation of my thesis. The existence of rental duration make the rental problem a multiple states problem. From reviewing the related literature, we know that multi states problems are very difficult problems in revenue management. The difficulty of the multi states problem is the interaction among the multi states. Rental Problem has common difficulty of the multi states problem. What's more it has the effects of the return process.

According to the characters of rental problem, we use two stages discrete time model to operate items return and price dynamically. We set up this rental problem as following two parts problem: a dynamic pricing problem in stage 2 and an inventory control problem in stage 1.

We shows the existence and uniqueness of the optimal pricing policy in rental problem in Proposition 1. The properties of rental problem in Proposition 1 guarantees the correctness of the numerical methods. We first use numerical method to analyze the effects of the potential market size and the price sensitive index. The numerical results show that, the bigger potential market size and smaller price sensitive index derive the greater optimal expected revenue, when all others coefficients are fixed. The numerical analysis also suggest the smaller potential market size and bigger price sensitive index result high inventory utilization rate. We then use numerical method to analyze the effect of inventory level. The numerical results suggest the optimal expected revenue is concave in the inventory level. We mention it in Remark 1. We also study the time properties of rental problem through numerical method, the numerical result show that the existence of rental duration breaks the time properties of the rental problem.

We solve the deterministic demand problem in Proposition 2. Proposition 2 provides the optimal solutions to both two case of deterministic demand problem: analysis periods K is divided evenly by rental duration r and analysis periods K isn't divided evenly by rental duration r. We also prove the revenue of deterministic demand is concave in the inventory level in Proposition 2. The optimal solution derived by Proposition 2 is the foundation of the periodical fixed price policy.

The deterministic demand problem provide a upper to the stochastic problem. Proposition 3 is very important to prove the asymptotical optimal of the PFP policy. In Proposition 4, we prove the PFP policy is asymptotical optimal as the demand and the inventory go infinity. The numerical simulation shows the PFP policy performance steady and close to the optimal policy.

Remark 1 is the first limitation of my thesis. We find the optimal expected revenue is concave in the inventory level through numerical method, but we just prove these concavity in the condition when rent out items can't return to the system. This return effect is the difficulty of the rental problem. If we can find another effect in rental problem dominate the return effect, the concavity holds.

The second limitation of my thesis is the computation complexity of our model is high. Its computation complexity exponential increase as the inventory increase and as the rental duration increase. But fortunately, we have the following heuristics to overcome this computation complexity. The first heuristics is the periodical fixed price policy. The second is the dynamical fixed priced policy, every day we use deterministic demand to decide the rental price.

The third limitation of our optimal stochastic price policy is that we multiple times changes the price in one day depend on the inventory and the left on hand inventory. In practice, the retailer will not trails the inventory level and change the rental price all the time. Like the taxi fee in Singapore has the peak time charge and free time charge, an more practical rental policy should consider both the cost of change price and benefit of it.

We have two directions of the future works. The first one is consider a company rent multi products to the customers, because in video rental business, no retailer will rent just one kind of videos to the customers.Even This multiple rental problem has both practical and theoretical value, the challenge of it is obviously. Another potential research problem is considering the supply contracts analysis of rental business. This is also an interesting problem to the suppliers of the rental retailers.

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Appendices

Appendix .A Proof of Proposition and Lemma

A very useful observation by Scarf (1958) [19] that the expected value of quadratic function is invariant. Thus if $G(x) = \alpha + \beta x + \gamma x^2$, then

$$\mathbb{E}G(x) = \int G(x)dF(x) = \alpha + \beta\mu + \gamma(\mu^2 + \sigma^2)$$

From this we could get the Proposition 1 in Gallego (1992) [9],

$$\int \max(x - R, 0)dF(x) = \frac{1}{2}(\sqrt{\sigma^2 + \Delta^2} - \Delta)$$

where $\Delta = R - \mu$. Which is straight forward to go to the equation (18) of Gallego and van Ryzin (1994) [10],

$$\mathbb{E}[(N-n)^+] \leq \frac{\sqrt{\sigma^2 + (n-\mu)^2} - n - \mu}{2},$$

where N is a random variable with finite mean μ and a standard deviation σ, and n is a real number.

Proof of Proposition 1.

We first show that the supremum in equation (2)can be replaced by $\max_{\lambda_K \in [0,\lambda^*]}$. To do so, let λ_i be an arbitrary intensity satisfying $\lambda_i > \lambda^*$. By the concavity of $r(\lambda)$ and the definition of of λ^*,we have $r(\lambda^*) > r(\lambda_i)$,and since $J^*_{k,t}(n, n_1, \ldots, n_r, n_{r+1})$ is nondecreasing in n, we have

$$\begin{aligned} r(\lambda^*) - \lambda^*[J_{k,t}(n, n_1, \ldots, n_r, n_{r+1}) - J_{k,t}(n-1, n_1, n_2, \ldots, n_{r-1})] &\geq \\ r(\lambda_i) - \lambda_i[J_{k,t}(n, n_1, \ldots, n_r, n_{r+1}) - J_{k,t}(n-1, n_1, n_2, \ldots, n_{r-1})] &. \end{aligned}$$

From equation (1), we know the optimal choice of λ is always within the set $[0, \lambda^*]$, a compact set. Combining compactness with the fact that $r(\lambda)$ is continuous and bounded establishes the conditions required by Bremaud Theorem II.3 for the existence of a unique solution to equation (1). □

Lemma 1. We use contradiction to prove lemma 1, if fixed price isn't optimal inside one period, which means at least two time interval has different prices, if we increase one price a very small price, decrease another price the same price, the total rental out items are the same, but because the revenue rate function is concave, so we get a better price policy, which contradict to the assumption, so the fixed price policy is the optimal policy inside one period. □

Proof of Proposition 2. If K is divided by r evenly, when the single price policy is not optimal, at least two period price is different, if we increase one price a very small price, decrease another price the same price, the total rental out items are the same, but because the revenue rate function is concave, so we get a better price policy, which contradict to the assumption, so the single price policy is the optimal policy in these situation.

If K isn't divided by r evenly, $(n+1)\Delta r_1(x) = n\Delta r_2(x')$ is not the optimal condition of optimal policy, at least two period don't satisfy these condition, we if we increase one rental out item a very small number, decrease another rent out items the same number, the total rental out items are the same, but because the revenue rate function is concave, so we get a better price policy, which contradict to the assumption, so the optimal condition is $(n+1)\Delta r_1(x) = n\Delta r_2(x')$.

The optimal revenue is easy to get following the way by Gallego and van Ryzin (1994).

When the rental information is $(n, n_1, \ldots, n_r)$, we assume the optimal rental number is $m_k, m_{k+1}, \ldots, m_K$.

If no return we need to consider, when we add one more unit to the inventory on hand, $(n+1, n_1, \ldots, n_r)$, the optimal policy is rent this item at the day with largest marginal profit, which means the optimal rental number is $m_k, m_{k+1}, \ldots, m_i + 1, \ldots, m_K$. We use contradiction to prove this point. If this is not true, there at least exist $m_k, m_{k+1}, \ldots, m_j - 1, \ldots, m_{j'} + 1, \ldots, m_i + 1, \ldots, m_K$ is a better policy. Which contradict that $m_k, m_{k+1}, \ldots, m_K$ is the optimal rental number when the rental information is $(n, n_1, \ldots, n_r)$ or $m_k, m_{k+1}, \ldots, m_j -$

$1,\dots,m_{j'}+2,\dots,m_i,\dots,m_K$, because $m_k,m_{k+1},\dots,m_j-1,\dots,m_{j'}+2,\dots,m_i,\dots,m_K \leq m_k,m_{k+1},\dots,m_j,\dots,m_{j'}+1,\dots,m_i,\dots,m_K$ which contradict m_i has the max marginal profit. So when we add one more unit to the inventory on hand, $(n+1,n_1,\dots,n_r)$, the optimal policy is rent this item at the day with largest marginal revenue, which means the optimal rental number is $m_k,m_{k+1},\dots,m_i+1,\dots,m_K$.

When we have another one more unit, which means the rental information is $(n+2,n_1,\dots,n_r)$, the optimal rental number is either $m_k,m_{k+1},\dots,m_i+2,\dots,m_K$ or $m_k,m_{k+1},\dots,m_j+1,\dots,m_i+1,\dots,m_K$. Because $r(\cdot)$ is a concave function and m_i has the maximized marginal revenue. So the marginal revenue of this item is smaller than the former one, which complete the proof of $\Pi_k^D(n,n_1,\dots,n_r)$ is concave in n.

When we consider the return process, given $(n,n_1,\dots,n_r)$, the optimal rental number is $m_k,m_{k+1},\dots,m_K$. When we add one more unit to the on hand inventory $(n+1,n_1,\dots,n_r)$, the optimal policy is allocating this unit to the days with largest marginal revenue. $\square$

Proof of Proposition 3. From Proposition 1, we have $\lambda_s \leq \lambda^*$, which implies $\sum_{s=t}^{T} < \infty$ for all $t \geq 1$. Recall $\mathscr{U}$ denotes the classes of policies which satisfy $\sum_{t=1}^{T} dN_t^K \leq n$; therefore by Bremaud (1980) [4], Theorem II,

$$\mathbb{E}_{u_K}\left[\sum_{s=t}^{T} dN_s^K\right] = \mathbb{E}_{u_K}\left[\sum_{s=t}^{T} \lambda_s^K\right] \leq n$$

Because we consider a Poisson Process, so the price at time interval s is a function of $p_s = p_s^{u_K}(n - N_s)$ only. By Bremaud (1980) [4], Theorem II, we can write,

$$\begin{aligned}\Pi_{K,t}^{u_k}(n, n_r) &= \mathbb{E}_{u_K}\left[\sum_{s=t}^{T} p_s^{u_K}(n - N_s) dN_s^K\right] \\ &\quad \mathbb{E}_{u_K}\left[\sum_{s=t}^{T} r(\lambda_s^{u_K})\right] \qquad (1)\end{aligned}$$

and

$$\Pi_{K,t}(n, n_r) = \sup_{u_K \in \mathscr{U}} \Pi_{K,t}^{u_k}(n, n_r)$$

Now for $\mu \geq 0$, we define augment cost function

$$\begin{aligned}\Pi_{K,t}^{u_k}(n, n_1, n_2, \ldots, n_r, \mu) &= \mathbb{E}_{u_K}\left[\sum_{s=t}^{T} (r(\lambda_s^{u_K}) - \mu \lambda_s^{u_K})\right] \\ &\quad + n\mu \geq \Pi_{K,t}^{u_k}(n, n_r), \qquad (2)\end{aligned}$$

and the augmented deterministic cost function

$$\Pi_{K,t}^{D}(n, n_1, n_2, \ldots, n_r, \mu) = \max_{\lambda(s)} \sum_{s=t}^{T} (r(\lambda(s)) - \mu \lambda(s)) + n\mu \qquad (3)$$

The left of this proof can be divided to two part. First we want to prove

$$\Pi_{K,t}^{u_k}(n, n_1, n_2, \ldots, n_r, \mu) \leq \Pi_{K,t}^{D}(n, n_1, n_2, \ldots, n_r, \mu)$$

From the maximizing of point-wise:

$$\begin{aligned}
\Pi^{u_k}_{K,t}(n,n_1,n_2,\ldots,n_r,\mu) &= \mathbb{E}_{u_K}\left[\sum_{s=t}^{T}(r(\lambda_s^{u_K})-\mu\lambda_s^{u_K})\right]+n\mu \\
&= \sum_{s=t}^{T}\mathbb{E}_{u_K}\left(r(\lambda_s^{u_K})-\mu\lambda_s^{u_K}\right)+n\mu \\
&\leq \sum_{s=t}^{T}\max_{\lambda_s^{u_K}}\{r(\lambda_s^{u_K})-\mu\lambda_s^{u_K}\}+n\mu \\
&= \max_{\lambda_s^{u_K}}\sum_{s=t}^{T}(r(\lambda_s^{u_K})-\mu\lambda_s^{u_K})+n\mu \\
&\doteq \Pi^{D}_{K,t}(n,n_1,n_2,\ldots,n_r,\mu) \qquad (4)
\end{aligned}$$

Because for all $u_K \in \mathscr{U}$ and all $\mu \geq 0$ we have the same property, so we have

$$\Pi_{K,t}(n,n_r) \leq \inf_{\mu\geq 0}\Pi^{D}_{K,t}(n,n_1,n_2,\ldots,n_r,\mu)$$

Because $\Pi^{D}_{K,t}(n,n_1,n_2,\ldots,n_r,\mu)$ is the optimal dual value of the following program:

$$\begin{aligned}
\Pi^{D}_{K,t}(n,n_r) &= \max_{\lambda(s)}\sum_{s=t}^{T}r(\lambda_s^{u_K}) \qquad (5) \\
\text{subject to} \quad & \sum_{s=t}^{T}\lambda_s^{u_K} \leq n \qquad (6)
\end{aligned}$$

From Karush-Kuhn-Tucker optimal (KKT) conditions, we have

$$\Pi^{D}_{K,t}(n,n_r) = \inf_{\mu\geq 0}\Pi^{D}_{K,t}(n,n_1,n_2,\ldots,n_r,\mu) = \Pi^{D}_{K,t}(n,n_1,n_2,\ldots,n_r,\mu^*)$$

$\square$

Proof of Proposition 4.

1. If K is divided evenly by r, under periodical fixed price policy, the problem can be consider as an K times fixed priced problem: while the price is fixed at p is

$$p\mathbb{E}[X_{\lambda(p)} - (X_{\lambda(p)} - N/r)^{+}] \tag{7}$$

From Gallego (1992) shows that for any random variable X with finite mean μ and finite standard deviation σ, and for any real number N/r,

$$\mathbb{E}[(X - N/r)^{+}] \leq \frac{\sqrt{\sigma^2 + (N/r - \mu)^2} - (N/r - \mu)}{2} \tag{8}$$

where $x^{+} \doteq \max(x, 0)$. When the inventory is scarce, which means $\lambda^*(T - t + 1)\delta t > n$, we use the run out price p^0, using inequality (15) in equation (14), and when price is p^0, $\mu = n = \sigma^2$, we obtain

$$\Pi_{1,1}^{PFP} \geq np^0(1 - \frac{1}{2\sqrt{N/r}}) = K * r^0(T\delta t)(1 - \frac{1}{2\sqrt{N/r}})$$

In the case when the inventory is sufficient, we price at p^*, we obtain

$$\begin{aligned}
& \Pi_{1,1}^{PFP} \\
\geq\ & K * p^* \Big(\lambda^* \cdot (T\delta t) \\
& - \frac{\sqrt{\lambda^* \cdot (T\delta t) + (n - \lambda^* \cdot (T\delta t))^2} - (n - \lambda^* \cdot (T\delta t))}{2}\Big) \\
\geq\ & K * p^* \lambda^* \cdot (T\delta t) \left(1 - \frac{1}{2\sqrt{\lambda^* \cdot (T\delta t)}}\right) \\
=\ & K * r^* \cdot (T\delta t) \left(1 - \frac{1}{2\sqrt{\lambda^* \cdot (T\delta t)}}\right)
\end{aligned}$$

Because $\Pi^D_{1,1} = K * (T\delta t)\min(r^0, r^*)$ and $\Pi^D_{1,1} \geq \Pi_{1,1}$, we have

$$\frac{\Pi^{PFP}_{1,1}(N,0,\ldots,0,0)}{\Pi_{1,1}(N,0,\ldots,0,0)} \geq 1 - \frac{1}{2\sqrt{\min\{N/r, \lambda^* \cdot (T\delta t)\}}}$$

2. When K isn't divided evenly by r, under periodical fixed price policy, the problem can be consider as an $(n+1)l$ times fixed priced problem plus another $n(r-l)$ times fixed priced problem : this is an two fixed price problem, which include $(n+1)l$ periods use price $p1$ and $n(r-l)$ periods use price $p2$, $p2 > p1$, and $x > x'$: use the same method in part 1, we can get

$$\frac{\Pi^{PFP}_{1,1}(N,0,\ldots,0,0)}{\Pi_{1,1}(N,0,\ldots,0,0)} \geq 1 - \frac{1}{2\sqrt{\min\{x', \lambda^* \cdot (T\delta t)\}}}$$

where x' satisfies $lx + (r-l)x' = N$ and $(n+1)\Delta r_1(x) = n\Delta r_2(x')$.

Because the expected revenue of the optimal stochastic policy is bigger than the periodical fixed price policy. We proof the result of Proposition 4.

$\square$

Appendix .B Proof of Remark

Proof of Remark 1.

1. The monotonicity of the optimal expected revenue $\Pi_{k,t}(n,n_1,\ldots,n_r,n_{r+1})$ is straightforward to show. We start from the last period. Because $\Pi_{K+1,1}(n,n_0,n_1,$ $n_2,\ldots,n_r)=0$, so $\Delta_{n_i}\Pi_{K,t}(n,n_1,\ldots,n_r,n_{r+1})$ equal to 0 for all n_i, $i=1,2,\ldots,r$. We first prove $\Delta_n\Pi_{K,t}(n,n_1,\ldots,n_r,n_{r+1})$ is decreasing in n. For simplification, we just use n to describe the last period state and the marginal expect revenue:

$$\begin{aligned}
&\Delta_n\Pi_{K,t}(n,n_1,\ldots,n_r,n_{r+1}) = \Delta_n\Pi_{K,t}(n,n_1,\ldots,n_r,n_{r+1})\\
&= \Pi_{K,t}(n,n_1,\ldots,n_r,n_{r+1}) - \Pi_{K,t}(n-1,n_1,\ldots,n_r,n_{r+1})
\end{aligned}$$

From the boundary condition $\Pi_{K,t+1}(n,n_1,\ldots,n_r,n_{r+1})=0$, then we have $\Delta_n\Pi_{K,t+1}($ $n+1,n_1,\ldots,n_r,n_{r+1})-\Delta_n\Pi_{K,t+1}(n,n_1,\ldots,n_r,n_{r+1})\leq 0$, we use induction to prove this proposition: assume $\Delta_n\Pi_{K,t+1}(n+1,n_1,\ldots,n_r,n_{r+1})-\Delta_n\Pi_{K,t+1}(n,n_1,\ldots,n_r,$ $n_{r+1})\leq 0$. For period t, from equation (3) we have

$$\begin{aligned}
&\Delta_n\Pi_{K,t}(n,n_1,\ldots,n_r,n_{r+1})\\
= \;& \Pi_{K,t}(n,n_1,\ldots,n_r,n_{r+1}) - \Pi_{K,t}(n-1,n_1,\ldots,n_r,n_{r+1})\\
= \;& r(\lambda^*_{K,t}(n,n_1,\ldots,n_r,n_{r+1}))\delta t + \lambda^*_{K,t}(n,n_1,\ldots,n_r,n_{r+1})\delta t\Pi_{K,t+1}(n-1,n_1,\ldots,n_r,n_{r+1})\\
&+(1-\lambda^*_{K,t}(n,n_1,\ldots,n_r,n_{r+1})\delta t)\Pi_{K,t+1}(n,n_1,\ldots,n_r,n_{r+1})\\
&-[r(\lambda^*_{K,t}(n-1,n_1,\ldots,n_r,n_{r+1}))\delta t + \lambda^*_{K,t}(n-1,n_1,\ldots,n_r,n_{r+1})\delta t\Pi_{K,t+1}(n-2)\\
&+(1-\lambda^*_{K,t}(n,n_1,\ldots,n_r,n_{r+1})\delta t)\Pi_{K,t+1}(n-1,n_1,\ldots,n_r,n_{r+1})]
\end{aligned}$$

Then we have

$$
\begin{aligned}
& \Delta_n\Pi_{K,t}(n+1,n_1,\ldots,n_r,n_{r+1}) - \Delta_n\Pi_{K,t}(n,n_1,\ldots,n_r,n_{r+1}) \\
= \; & r(\lambda^*_{K,t}(n+1,n_1,\ldots,n_r,n_{r+1}))\delta t + \lambda^*_{K,t}(n+1,n_1,\ldots,n_r,n_{r+1})\delta t\Pi_{K,t+1}(n,n_1,\ldots,n_r,n_{r+1}) \\
& +(1-\lambda^*_{K,t}(n+1,n_1,\ldots,n_r,n_{r+1})\delta t)\Pi_{K,t+1}(n+1,n_1,\ldots,n_r,n_{r+1}) \\
& -[r(\lambda^*_{K,t}(n,n_1,\ldots,n_r,n_{r+1}))\delta t + \lambda^*_{K,t}(n,n_1,\ldots,n_r,n_{r+1})\delta t\Pi_{K,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
& +(1-\lambda^*_{K,t}(n,n_1,\ldots,n_r,n_{r+1})\delta t)\Pi_{K,t+1}(n,n_1,\ldots,n_r,n_{r+1})] \\
& -[r(\lambda^*_{K,t}(n,n_1,\ldots,n_r,n_{r+1}))\delta t + \lambda^*_{K,t}(n,n_1,\ldots,n_r,n_{r+1})\delta t\Pi_{K,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
& +(1-\lambda^*_{K,t}(n,n_1,\ldots,n_r,n_{r+1})\delta t)\Pi_{K,t+1}(n,n_1,\ldots,n_r,n_{r+1})] \qquad (P0) \\
& +[r(\lambda^*_{K,t}(n-1,n_1,\ldots,n_r,n_{r+1}))\delta t + \lambda^*_{K,t}(n-1,n_1,\ldots,n_r,n_{r+1})\delta t\Pi_{K,t+1}(n-2) \\
& +(1-\lambda^*_{K,t}(n,n_1,\ldots,n_r,n_{r+1})\delta t)\Pi_{K,t+1}(n-1,n_1,\ldots,n_r,n_{r+1})]
\end{aligned}
$$

From the definition of $\lambda^*K,t(n,n_1,\ldots,n_r,n_{r+1})$, we get that

$$
\begin{aligned}
& r(\lambda^*_{K,t}(n,n_1,\ldots,n_r,n_{r+1}))\delta t + \lambda^*_{K,t}(n,n_1,\ldots,n_r,n_{r+1})\delta t\Pi_{K,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
& +(1-\lambda^*_{K,t}(n,n_1,\ldots,n_r,n_{r+1})\delta t)\Pi_{K,t+1}(n,n_1,\ldots,n_r,n_{r+1}) \\
\geq \; & r(\lambda^*_{K,t}(n+1,n_1,\ldots,n_r,n_{r+1}))\delta t + \lambda^*_{K,t}(n+1,n_1,\ldots,n_r,n_{r+1})\delta t\Pi_{K,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
& +(1-\lambda^*_{K,t}(n+1,n_1,\ldots,n_r,n_{r+1})\delta t)\Pi_{K,t+1}(n,n_1,\ldots,n_r,n_{r+1})
\end{aligned}
$$

and

$$
\begin{aligned}
& r(\lambda^*_{K,t}(n,n_1,\ldots,n_r,n_{r+1}))\delta t + \lambda^*_{K,t}(n,n_1,\ldots,n_r,n_{r+1})\delta t\Pi_{K,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
& +(1-\lambda^*_{K,t}(n,n_1,\ldots,n_r,n_{r+1})\delta t)\Pi_{K,t+1}(n,n_1,\ldots,n_r,n_{r+1}) \\
\geq \; & r(\lambda^*_{K,t}(n-1,n_1,\ldots,n_r,n_{r+1}))\delta t + \lambda^*_{K,t}(n-1,n_1,\ldots,n_r,n_{r+1})\delta t\Pi_{K,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
& +(1-\lambda^*_{K,t}(n-1,n_1,\ldots,n_r,n_{r+1})\delta t)\Pi_{K,t+1}(n,n_1,\ldots,n_r,n_{r+1})
\end{aligned}
$$

Substituting into $(P0)$, we can obtain:

$$\begin{aligned}
& \Delta_n\Pi_{K,t}(n+1,n_1,\dots,n_r,n_{r+1}) - \Delta_n\Pi_{K,t}(n,n_1,\dots,n_r,n_{r+1}) \\
\leq\ & r(\lambda^*_{K,t}(n+1,n_1,\dots,n_r,n_{r+1}))\delta t + \lambda^*_{K,t}(n+1,n_1,\dots,n_r,n_{r+1})\delta t\Pi_{K,t+1}(n,n_1,\dots,n_r,n_{r+1}) \\
& +(1-\lambda^*_{K,t}(n+1,n_1,\dots,n_r,n_{r+1})\delta t)\Pi_{K,t+1}(n+1,n_1,\dots,n_r,n_{r+1}) \\
& -[r(\lambda^*_{K,t}(n+1,n_1,\dots,n_r,n_{r+1}))\delta t \\
& +\lambda^*_{K,t}(n+1,n_1,\dots,n_r,n_{r+1})\delta t\Pi_{K,t+1}(n-1,n_1,\dots,n_r,n_{r+1}) \\
& +(1-\lambda^*_{K,t}(n+1,n_1,\dots,n_r,n_{r+1})\delta t)\Pi_{K,t+1}(n,n_1,\dots,n_r,n_{r+1})] \\
& -[r(\lambda^*_{K,t}(n-1,n_1,\dots,n_r,n_{r+1}))\delta t \\
& +\lambda^*_{K,t}(n-1,n_1,\dots,n_r,n_{r+1})\delta t\Pi_{K,t+1}(n-1,n_1,\dots,n_r,n_{r+1}) \\
& +(1-\lambda^*_{K,t}(n-1,n_1,\dots,n_r,n_{r+1})\delta t)\Pi_{K,t+1}(n,n_1,\dots,n_r,n_{r+1})] \\
& +[r(\lambda^*_{K,t}(n-1,n_1,\dots,n_r,n_{r+1}))\delta t + \lambda^*_{K,t}(n-1,n_1,\dots,n_r,n_{r+1})\delta t\Pi_{K,t+1}(n-2) \\
& +(1-\lambda^*_{K,t}(n-1,n_1,\dots,n_r,n_{r+1})\delta t)\Pi_{K,t+1}(n-1,n_1,\dots,n_r,n_{r+1})] \\
=\ & \lambda^*_{K,t}(n+1,n_1,\dots,n_r,n_{r+1})\delta t[\Delta\Pi_{K,t+1}(n,n_1,\dots,n_r,n_{r+1})] \\
& +(1-\lambda^*_{K,t}(n+1,n_1,\dots,n_r,n_{r+1})\delta t)[\Delta\Pi_{K,t+1}(n+1,n_1,\dots,n_r,n_{r+1})] \\
& -\lambda^*_{K,t}(n-1,n_1,\dots,n_r,n_{r+1})\delta t[\Delta\Pi_{K,t+1}(n-1,n_1,\dots,n_r,n_{r+1})] \\
& -(1-\lambda^*_{K,t}(n-1,n_1,\dots,n_r,n_{r+1})\delta t)[\Delta\Pi_{K,t+1}(n,n_1,\dots,n_r,n_{r+1})] \\
=\ & -\lambda^*_{K,t}(n+1,n_1,\dots,n_r,n_{r+1})\delta t[\Delta\Pi_{K,t+1}(n+1,n_1,\dots,n_r,n_{r+1}) \\
& -\Delta_n\Pi_{K,t+1}(n,n_1,\dots,n_r,n_{r+1})] + \Delta_n\Pi_{K,t+1}(n+1,n_1,\dots,n_r,n_{r+1}) \\
& +\lambda^*_{K,t}(n-1,n_1,\dots,n_r,n_{r+1})\delta t[\Delta_n\Pi_{K,t+1}(n,n_1,\dots,n_r,n_{r+1}) \\
& -\Delta_n\Pi_{K,t+1}(n-1,n_1,\dots,n_r,n_{r+1})] - \Delta_n\Pi_{K,t+1}(n,n_1,\dots,n_r,n_{r+1}) \\
=\ & (1-\lambda^*_{K,t}(n+1,n_1,\dots,n_r,n_{r+1}))[\Delta_n\Pi_{K,t+1}(n+1,n_1,\dots,n_r,n_{r+1}) \\
& -\Delta\Pi_{K,t+1}(n,n_1,\dots,n_r,n_{r+1})] \\
& +\lambda^*_{K,t}(n-1,n_1,\dots,n_r,n_{r+1})\delta t[\Delta_n\Pi_{K,t+1}(n,n_1,\dots,n_r,n_{r+1}) \\
& -\Delta_n\Pi_{K,t+1}(n-1,n_1,\dots,n_r,n_{r+1})]
\end{aligned}$$

By induction $\Delta_n\Pi_{K,t+1}(n+1,n_1,\ldots,n_r,n_{r+1}) - \Delta_n\Pi_{K,t+1}(n,n_1,\ldots,n_r,n_{r+1}) \le 0$ and $\Delta_n\Pi_{K,t+1}(n,n_1,\ldots,n_r,n_{r+1}) - \Delta_n\Pi_{K,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}) \le 0$, and by definition $0 \le \lambda^*K,t(n+1,n_1,\ldots,n_r,n_{r+1})\delta t \le 1$. So we have $\Delta_n\Pi_{K,t}(n+1,n_1,\ldots,n_r, n_{r+1}) - \Delta_n\Pi_{K,t}(n,n_1,\ldots,n_r,n_{r+1}) \le 0$, this complete the proof for last period.

2. For the last second period, we just need to use the two variable n and n_r to describe the state, and because $\Pi_{K-1,T+1}(n,n_r) = \Pi_{K,1}(n+n_r)$,

$$\begin{aligned}
\Delta_n\Pi_{K-1,T+1}(n,n_r) &= \Pi_{K-1,T+1}(n,n_r) \\
&\quad -\Pi_{K-1,T+1}(n-1,n_r) \\
&= \Pi_{K,1}(n+n_r) \\
&\quad -\Pi_{K,1}(n+n_r-1) \\
&= \Delta_n\Pi_{K,1}(n+n_r)
\end{aligned}$$

and $\Delta_n\Pi_{K-1,T+1}(n-1,n_r) = \Delta_n\Pi_{K,1}(n+n_r-1)$. From part 1, we know $\Delta_n\Pi_{K-1,T+1}(n,n_r) \le \Delta_n\Pi_{K-1,T+1}(n-1,n_r)$. Following the same methodology in Part 1, we can prove $\Delta_n\Pi_{K-1,t}(n,n_r) \le \Delta_n\Pi_{K-1,t}(n-1,n_r)$ for all $t = 1,2,\ldots,t$.

3. Use the same method, we can prove $\Delta_n\Pi_{K-r,t}(n,n_1,n_2,\ldots,n_r) \le \Delta_n\Pi_{K-r,t}(n-1,n_1,n_2,\ldots,n_r)$ in day $K-r$. For day $K-(r+1)$, we have

$$\begin{aligned}
&\Delta_n\Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) \\
=\ &\Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) - \Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1})
\end{aligned}$$

For period $T+1$, use the boundary condition

$$
\begin{aligned}
& \Delta_n \Pi_{K-(r+1),T+1}(n,n_1,\ldots,n_r,n_{r+1}) \\
= \; & \Pi_{K-(r+1),T+1}(n,n_1,\ldots,n_r,n_{r+1}) - \Pi_{K-(r+1),T+1}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
= \; & \Pi_{K-r,1}(n+n_r,0,n_0,n_1,\ldots,n_{r-1}) - \Pi_{K-r,1}(n+n_r-1,0,n_0,n_1,\ldots,n_{r-1}) \\
= \; & \Delta_n \Pi_{K-r,1}(n+n_r,0,n_0,n_1,\ldots,n_{r-1})
\end{aligned}
$$

and we can get $\Delta_n \Pi_{K-(r+1),T+1}(n+1,n_1,\ldots,n_r,n_{r+1}) \leq \Delta_n \Pi_{K-(r+1),T+1}(n,n_1,\ldots,n_r,n_{r+1})$.

From equation (2), in period $K-(r+1)$, we have

$$
\begin{aligned}
& \Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) \\
= \; & \sup_{\lambda}[\lambda \delta t(p(\lambda)+\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)) \\
& +(1-\lambda \delta t)\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1}) \\
= \; & r(\lambda^*_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}))\delta t \\
& +\lambda^*_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1})\delta t \Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1) \\
& +(1-\lambda^*_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1})\delta t)\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1})
\end{aligned}
$$

So

$$
\begin{aligned}
& \Delta_n \Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) \\
= \; & \Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) - \Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
= \; & r(\lambda^*_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}))\delta t \\
& +\lambda^*_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1})\delta t \Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1) \\
& +(1-\lambda^*_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1})\delta t)\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1}) \\
& -[r(\lambda^*_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}))\delta t \\
& +\lambda^*_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1})\delta t \Pi_{K-(r+1),t+1}(n-2,n_0+1,n_1,n_2,\ldots,n_r) \\
& +(1-\lambda^*_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1})\delta t)\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1})]
\end{aligned}
$$

and

$$
\begin{aligned}
& \Delta_n \Pi_{K-r,t}(n+1, n_1, \ldots, n_r, n_{r+1}) - \Delta_n \Pi_{K-(r+1),t}(n, n_1, \ldots, n_r, n_{r+1}) \\
= \; & r(\lambda^*_{K-(r+1),t}(n+1, n_1, \ldots, n_r, n_{r+1}))\delta t \\
& + \lambda^*_{K-(r+1),t}(n+1, n_1, \ldots, n_r, n_{r+1})\delta t \Pi_{K-(r+1),t+1}(n, n_0+1, n_1, n_2, \ldots, n_r) \\
& + (1 - \lambda^*_{K-(r+1),t}(n+1, n_1, \ldots, n_r, n_{r+1})\delta t)\Pi_{K-(r+1),t+1}(n+1, n_1, \ldots, n_r, n_{r+1}) \\
& - [r(\lambda^*_{K-(r+1),t}(n, n_1, \ldots, n_r, n_{r+1}))\delta t \\
& + \lambda^*_{K-(r+1),t}(n, n_1, \ldots, n_r, n_{r+1})\delta t \Pi_{K-(r+1),t+1}(n-1, n_1, \ldots, n_r, n_{r+1}+1) \\
& + (1 - \lambda^*_{K-(r+1),t}(n, n_1, \ldots, n_r, n_{r+1})\delta t)\Pi_{K-(r+1),t+1}(n, n_1, \ldots, n_r, n_{r+1})] \\
& - [r(\lambda^*_{K-(r+1),t}(n, n_1, \ldots, n_r, n_{r+1}))\delta t \\
& + \lambda^*_{K-(r+1),t}(n, n_1, \ldots, n_r, n_{r+1})\delta t \Pi_{K-(r+1),t+1}(n-1, n_1, \ldots, n_r, n_{r+1}+1) \\
& + (1 - \lambda^*_{K-(r+1),t}(n, n_1, \ldots, n_r, n_{r+1})\delta t)\Pi_{K-(r+1),t+1}(n, n_1, \ldots, n_r, n_{r+1})] \\
& + [r(\lambda^*_{K-(r+1),t}(n-1, n_1, \ldots, n_r, n_{r+1}))\delta t \\
& + \lambda^*_{K-(r+1),t}(n-1, n_1, \ldots, n_r, n_{r+1})\delta t \Pi_{K-(r+1),t+1}(n-2, n_0+1, n_1, n_2, \ldots, n_r) \\
& + (1 - \lambda^*_{K-(r+1),t}(n-1, n_1, \ldots, n_r, n_{r+1})\delta t)\Pi_{K-(r+1),t+1}(n-1, n_1, \ldots, n_r, n_{r+1})] \\
\leq \; & \lambda^*_{K-(r+1),t}(n+1, n_1, \ldots, n_r, n_{r+1})\delta t \Pi_{K-(r+1),t+1}(n, n_0+1, n_1, n_2, \ldots, n_r) \\
& + (1 - \lambda^*_{K-(r+1),t}(n+1, n_1, \ldots, n_r, n_{r+1})\delta t)\Pi_{K-(r+1),t+1}(n+1, n_1, \ldots, n_r, n_{r+1}) \\
& - [\lambda^*_{K-(r+1),t}(n-1, n_1, \ldots, n_r, n_{r+1})\delta t \Pi_{K-(r+1),t+1}(n-1, n_1, \ldots, n_r, n_{r+1}+1)
\end{aligned}
$$

$$
\begin{aligned}
& +(1-\lambda^*_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1})\delta t)\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1})] \\
& -[\lambda^*_{K-(r+1),t}(n+1,n_1,\ldots,n_r,n_{r+1})\delta t\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1) \\
& +(1-\lambda^*_{K-(r+1),t}(n+1,n_1,\ldots,n_r,n_{r+1})\delta t)\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1})] \\
& +[\lambda^*_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1})\delta t\Pi_{K-(r+1),t+1}(n-2,n_0+1,n_1,n_2,\ldots,n_r) \\
& +(1-\lambda^*_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1})\delta t)\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1})] \\
= \; & \lambda^*_{K-(r+1),t}(n+1,n_1,\ldots,n_r,n_{r+1})\delta t\Delta_n\Pi_{K-(r+1),t+1}(n,n_0+1,n_1,n_2,\ldots,n_r) \\
& +(1-\lambda^*_{K-(r+1),t}(n+1,n_1,\ldots,n_r,n_{r+1})\delta t)\Delta_n\Pi_{K-(r+1),t+1}(n+1,n_1,\ldots,n_r,n_{r+1}) \\
& -\lambda^*_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1})\delta t\Delta_n\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1) \\
& -(1-\lambda^*_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1})\delta t)\Delta_n\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1}) \\
= \; & -\lambda^*_{K-(r+1),t}(n+1,n_1,\ldots,n_r,n_{r+1})\delta t \\
& [\Delta_n\Pi_{K-(r+1),t+1}(n+1,n_1,\ldots,n_r,n_{r+1})-\Delta_n\Pi_{K-(r+1),t+1}(n,n_0+1,n_1,n_2,\ldots,n_r)] \\
& +\Delta_n\Pi_{K-(r+1),t+1}(n+1,n_1,\ldots,n_r,n_{r+1}) \\
& +\lambda^*_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1})\delta t \\
& [\Delta_n\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1})-\Delta_n\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)] \\
& -\Delta_n\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1}) \\
\leq \; & -\lambda^*_{K-(r+1),t}(n+1,n_1,\ldots,n_r,n_{r+1})\delta t \\
& [\Delta_n\Pi_{K-(r+1),t+1}(n+1,n_1,\ldots,n_r,n_{r+1})-\Delta_n\Pi_{K-(r+1),t+1}(n,n_0+1,n_1,n_2,\ldots,n_r)] \\
& +\lambda^*_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1})\delta t \\
& [\Delta_n\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1})-\Delta_n\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)] \\
& +[\Delta_n\Pi_{K-(r+1),t+1}(n+1,n_1,\ldots,n_r,n_{r+1})-\Delta_n\Pi_{K-(r+1),t+1}(n,n_0+1,n_1,n_2,\ldots,n_r)] \\
= \; & (1-\lambda^*_{K-(r+1),t}(n+1,n_1,\ldots,n_r,n_{r+1})\delta t) \\
& [\Delta_n\Pi_{K-(r+1),t+1}(n+1,n_1,\ldots,n_r,n_{r+1})-\Delta_n\Pi_{K-(r+1),t+1}(n,n_0+1,n_1,n_2,\ldots,n_r)] \\
& +\lambda^*_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1})\delta t \\
& [\Delta_n\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1})-\Delta_n\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)]
\end{aligned}
$$

The last inequality comes from

$$\Delta_n \Pi_{K-(r+1),t+1}(n, n_0+1, n_1, n_2, \ldots, n_r) \leq \Delta_n \Pi_{K-(r+1),t+1}(n, n_0, n_1, n_2, \ldots, n_r)$$

So the key point is to prove remark 1:

Proof of Remark 1.

Before we prove remark 1, we define $\Delta\Pi_{k,t}(n, n_1, \ldots, n_r, n_{r+1})$ as follow,

$$\Delta\Pi_{k,t}(n, n_1, \ldots, n_r, n_{r+1}) = \Pi_{k,t}(n, n_1, \ldots, n_r, n_{r+1}) - \Pi_{k,t}(n-1, n_1, \ldots, n_r, n_{r+1}+1).$$

Because

$$\begin{aligned}
& \Delta_n \Pi_{K-(r+1),t}(n, n_1, \ldots, n_r, n_{r+1}) - \Delta_n \Pi_{K-(r+1),t}(n-1, n_1, \ldots, n_r, n_{r+1}+1) \\
= \; & \Pi_{K-(r+1),t}(n, n_1, \ldots, n_r, n_{r+1}) - \Pi_{K-(r+1),t}(n-1, n_1, \ldots, n_r, n_{r+1}) \\
& -(\Pi_{K-(r+1),t}(n-1, n_1, \ldots, n_r, n_{r+1}+1) - \Pi_{K-(r+1),t}(n-2, n_0+1, n_1, n_2, \ldots, n_r)) \\
= \; & \Pi_{K-(r+1),t}(n, n_1, \ldots, n_r, n_{r+1}) - \Pi_{K-(r+1),t}(n-1, n_1, \ldots, n_r, n_{r+1}+1) \\
& -(\Pi_{K-(r+1),t}(n-1, n_1, \ldots, n_r, n_{r+1}) - \Pi_{K-(r+1),t}(n-2, n_0+1, n_1, n_2, \ldots, n_r)) \\
= \; & \Delta\Pi_{K-(r+1),t}(n, n_1, \ldots, n_r, n_{r+1}) - \Delta\Pi_{K-(r+1),t}(n-1, n_1, \ldots, n_r, n_{r+1})
\end{aligned}$$

From equation (3),

$$\begin{aligned}
\Pi_{K-(r+1),t}(n, n_1, \ldots, n_r, n_{r+1}) \;\; = \;\; & \sup_{\lambda}[\lambda \delta t(p(\lambda) + \Pi_{K-r,t+1}(n-1, n_1, \ldots, n_r, n_{r+1}+1) \\
& +(1-\lambda \delta t)\Pi_{K-(r+1),t+1}(n, n_1, \ldots, n_r, n_{r+1})]
\end{aligned}$$

We have

$$
\begin{aligned}
& \Delta_n\Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) - \Delta_n\Pi_{k,t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) \\
= \; & \Delta\Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) - \Delta\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
= \; & \Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) - \Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) \\
& -(\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) - \Pi_{K-(r+1),t}(n-2,n_0+1,n_1,n_2,\ldots,n_r)) \\
= \; & [\lambda^*_{K-r,t}(n,n_1,\ldots,n_r,n_{r+1})\delta t(p(\lambda^*_{K-r,t}(n,n_1,\ldots,n_r,n_{r+1})) \\
& -\Delta\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1})) + \Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1})] \\
& -[\lambda^*_{K-r,t}(n-1,n_1,\ldots,n_r,n_{r+1}+1)\delta t(p(\lambda^*_{K-r,t}(n-1,n_1,\ldots,n_r,n_{r+1}+1)) \\
& -\Delta\Pi_{K-r,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)) + \Pi_{K-r,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)] \\
& -[\lambda^*_{K-r,t}(n-1,n_1,\ldots,n_r,n_{r+1})\delta t(p(\lambda^*_{K-r,t}(n-1,n_1,\ldots,n_r,n_{r+1})) \\
& -\Delta\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1})) + \Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1})] \\
& +[\lambda^*_{K-r,t}(n-2,n_0+1,n_1,n_2,\ldots,n_r)\delta t(p(\lambda^*_{K-r,t}(n-2,n_0+1,n_1,n_2,\ldots,n_r)) \\
& -\Delta\Pi_{K-r,t+1}(n-2,n_0+1,n_1,n_2,\ldots,n_r)) + \Pi_{K-r,t+1}(n-2,n_0+1,n_1,n_2,\ldots,n_r)] \\
= \; & [\lambda^*_{K-r,t}(n,n_1,\ldots,n_r,n_{r+1})\delta t(p(\lambda^*_{K-r,t}(n,n_1,\ldots,n_r,n_{r+1})) \\
& -\Delta\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1})) + \Delta\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1})] \\
& -[\lambda^*_{K-r,t}(n-1,n_1,\ldots,n_r,n_{r+1}+1)\delta t(p(\lambda^*_{K-r,t}(n-1,n_1,\ldots,n_r,n_{r+1}+1)) \\
& -\Delta\Pi_{K-r,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1))] \\
& -[\lambda^*_{K-r,t}(n-1,n_1,\ldots,n_r,n_{r+1})\delta t(p(\lambda^*_{K-r,t}(n-1,n_1,\ldots,n_r,n_{r+1})) \\
& -\Delta\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1})) + \Delta\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1})] \\
& +[\lambda^*_{K-r,t}(n-2,n_0+1,n_1,n_2,\ldots,n_r)\delta t(p(\lambda^*_{K-r,t}(n-2,n_0+1,n_1,n_2,\ldots,n_r)) \\
& -\Delta\Pi_{K-r,t+1}(n-2,n_0+1,n_1,n_2,\ldots,n_r))] \\
\leq \; & [\lambda^*_{K-r,t}(n,n_1,\ldots,n_r,n_{r+1})\delta t(p(\lambda^*_{K-r,t}(n,n_1,\ldots,n_r,n_{r+1})) \\
& -\Delta\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1})) + \Delta\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1})] \\
& -[\lambda^*_{K-r,t}(n-2,n_0+1,n_1,n_2,\ldots,n_r)\delta t(p(\lambda^*_{K-r,t}(n-2,n_0+1,n_1,n_2,\ldots,n_r)) \\
& -\Delta\Pi_{K-r,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1))]
\end{aligned}
$$

$$
\begin{aligned}
&-[\lambda^*_{K-r,t}(n,n_1,\ldots,n_r,n_{r+1})\delta t(p(\lambda^*_{K-r,t}(n,n_1,\ldots,n_r,n_{r+1}))\\
&-\Delta\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}))+\Delta\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1})]\\
&+[\lambda^*_{K-r,t}(n-2,n_0+1,n_1,n_2,\ldots,n_r)\delta t(p(\lambda^*_{K-r,t}(n-2,n_0+1,n_1,n_2,\ldots,n_r))\\
&-\Delta\Pi_{K-r,t+1}(n-2,n_0+1,n_1,n_2,\ldots,n_r))]\\
=\;&(1-\lambda^*_{K-r,t}(n,n_1,\ldots,n_r,n_{r+1})\delta t)(\Delta\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1})\\
&-\Delta\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}))\\
&+\lambda^*_{K-r,t}(n-2,n_0+1,n_1,n_2,\ldots,n_r)\delta t(\Delta\Pi_{K-r,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)\\
&-\Delta\Pi_{K-r,t+1}(n-2,n_0+1,n_1,n_2,\ldots,n_r))
\end{aligned}
$$

So the problem change to prove $\Delta\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1})-\Delta\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1})\leq 0$. Using induction, prove $\Delta\Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1})-\Delta\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1})\leq 0$ equals to prove $\Delta\Pi_{K-(r+1),T+1}(n,n_1,\ldots,n_r,n_{r+1})-\Delta\Pi_{K-(r+1),T+1}(n-1,n_1,\ldots,n_r,n_{r+1})\leq 0$, and because

$$
\begin{aligned}
&\Delta_n\Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1})-\Delta_n\Pi_{k,t}(n-1,n_1,\ldots,n_r,n_{r+1}+1)\\
=\;&\Delta\Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1})-\Delta\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1})
\end{aligned}
$$

So we just need to prove

$$
\Delta_n\Pi_{K-(r+1),T+1}(n,n_1,\ldots,n_r,n_{r+1})-\Delta_n\Pi_{K-r,T+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)\leq 0
$$

An important observation is $\Delta_n\Pi_{K-r,t}(n,n_1,\ldots,n_r,n_{r+1})=\Delta\Pi_{K-r,t}(n,n_1,\ldots,n_r,n_{r+1})$, until period $K-(r+1)$, so

$$\begin{aligned}
& \Delta_n \Pi_{K-(r+1),T+1}(n, n_1, \ldots, n_r, n_{r+1}) - \Delta_n \Pi_{K-r,T+1}(n-1, n_1, \ldots, n_r, n_{r+1}+1) \\
= \; & \Delta_n \Pi_{K-r,1}(n+n_r, 0, n_0, n_1, \ldots, n_{r-1}) - \Delta_n \Pi_{K-r,1}(n+n_r-1, 0, n_0+1, n_1, \ldots, n_{r-1}) \\
= \; & \Delta \Pi_{K-r,1}(n+n_r, 0, n_0, n_1, \ldots, n_{r-1}) - \Delta \Pi_{K-r,1}(n+n_r-1, 0, n_0+1, n_1, \ldots, n_{r-1})
\end{aligned}$$

Prove $\Delta\Pi_{K-r,1}(n+n_r, 0, n_0, n_1, \ldots, n_{r-1}) - \Delta\Pi_{K-r,1}(n+n_r-1, 0, n_0+1, n_1, \ldots, n_{r-1}) \leq 0$ is equivalent to prove $\Delta\Pi_{K-r+1,T+1}(n+n_r, 0, n_0, n_1, \ldots, n_{r-1}) - \Delta\Pi_{K-r+1,T+1}(n+n_r-1, 0, n_0+1, n_1, \ldots, n_{r-1}) \leq 0$, so we have,

$$\begin{aligned}
& \Delta \Pi_{K-r+1,T+1}(n+n_r, 0, n_0, n_1, \ldots, n_{r-1}) \\
& -\Delta \Pi_{K-r+1,T+1}(n+n_r-1, 0, n_0+1, n_1, \ldots, n_{r-1}) \leq 0 \\
\Leftrightarrow \; & \Delta_n \Pi_{K-r+1,T+1}(n+n_r, 0, n_0, n_1, \ldots, n_{r-1}) \\
& -\Delta_n \Pi_{K-r+1,T+1}(n+n_r-1, 0, n_0+1, n_1, \ldots, n_{r-1}) \leq 0 \\
\Leftrightarrow \; & \Delta_n \Pi_{K-r+2,1}(n+n_{r-1}+n_r, 0, 0, n_0, \ldots, n_{r-2}) \\
& -\Delta_n \Pi_{K-r+2,1}(n+n_{r-1}+n_r-1, 0, 0, n_0+1, \ldots, n_{r-2}) \leq 0
\end{aligned}$$

Use the same methodology we can prove in period $K-r$,

$$\Delta \Pi_{K-r,t}(n+1, n_1, \ldots, n_r, n_{r+1}) \leq \Delta \Pi_{K-r,t}(n, n_0+1, n_1, n_2, \ldots, n_r)$$

Then we want to prove

$$\Delta_n \Pi_{K-r,t}(n, n_1, \ldots, n_r, n_{r+1}) - \Delta_n \Pi_{K-r,t}(n, n_0+1, n_1, n_2, \ldots, n_r) \leq 0$$

□

We will give another way to get K1 and K2:

$$\begin{aligned}
& \Delta_n \Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) \\
= \; & \Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) - \Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
= \; & \Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) - \Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) \\
& +\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) - \Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
= \; & \Delta\Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) + \Delta_{n_{r+1}}\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1)
\end{aligned}$$

Then we can get

$$\begin{aligned}
& \Delta_n \Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) - \Delta_n \Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
= \; & \Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) - \Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
& \Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) - \Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}) \\
= \; & \Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) - \Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) \\
& +\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) - \Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
& -(\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) - \Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+1) \\
& +\Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+1) - \Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1})) \\
= \; & \Delta\Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) + \Delta_{n_{r+1}}\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) \\
& -(\Delta\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) + \Delta_{n_{r+1}}\Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+1)) \\
= \; & \Delta\Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) - \Delta\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
& +(\Delta_{n_{r+1}}\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) - \Delta_{n_{r+1}}\Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+1))
\end{aligned}$$

The last equation means if K1 and K2 exist, We prove our marginal inventory policy. So we try to prove the following two inequalities: $\Delta\Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) -$

$\Delta\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) \leq 0$ (K1) and $\Delta_{n_{r+1}}\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) - \Delta_{n_{r+1}}\Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+1) \leq 0$ (K2).

Then we prove prove K1: because $\Delta\Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) - \Delta\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1})$ satisfies $\Delta\Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) - \Delta\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) = \Delta_n\Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}+1) - \Delta_n\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1)$, which comes from

$$
\begin{aligned}
& \Delta\Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) - \Delta\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
= \; & \Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) - \Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) \\
& -(\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) - \Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+1) \\
= \; & \Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) - \Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
& -(\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) - \Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+1)) \\
= \; & \Delta_n\Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) - \Delta_n\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1)
\end{aligned}
$$

From the processing function

$$
\begin{aligned}
\Pi_{k,t}(n,n_1,\ldots,n_r,n_{r+1}) \; = \; & \sup_{\lambda}[\lambda\delta t(p(\lambda) + \Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)) \\
& +(1-\lambda\delta t)\Pi_{k,t+1}(n,n_1,\ldots,n_r,n_{r+1}) + o(\delta t)]
\end{aligned}
$$

we know that

$$
\begin{aligned}
& \Delta\Pi_{k,t}(n,n_1,\ldots,n_r,n_{r+1}) \\
= \; & \Pi_{k,t}(n,n_1,\ldots,n_r,n_{r+1}) - \Pi_{k,t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) \\
= \; & \lambda_1\delta t(p(\lambda_1) + \Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)) + (1-\lambda_1\delta t)\Pi_{k,t+1}(n,n_1,\ldots,n_r,n_{r+1}) \\
& -(\lambda_2\delta t(p(\lambda_2) + \Pi_{k,t+1}(n-2,n_1,\ldots,n_r,n_{r+1}+1)) + (1-\lambda_2\delta t)\Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}))
\end{aligned}
$$

and then we have

$$
\begin{aligned}
& \Delta\Pi_{k,t}(n,n_1,\ldots,n_r,n_{r+1}) - \Delta\Pi_{k,t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) \\
= \; & \Pi_{k,t}(n,n_1,\ldots,n_r,n_{r+1}) - \Pi_{k,t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) \\
& -(\Pi_{k,t}(n-1,n_1,\ldots,n_r,n_{r+1}) - \Pi_{k,t}(n-2,n_1,\ldots,n_r,n_{r+1}+1)) \\
= \; & \lambda_1\delta t(p(\lambda_1) + \Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)) + (1-\lambda_1\delta t)\Pi_{k,t+1}(n,n_1,\ldots,n_r,n_{r+1}) \\
& -(\lambda_2\delta t(p(\lambda_2) + \Pi_{k,t+1}(n-2,n_1,\ldots,n_r,n_{r+1}+2)) + (1-\lambda_2\delta t)\Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)) \\
& -(\lambda_3\delta t(p(\lambda_2) + \Pi_{k,t+1}(n-2,n_1,\ldots,n_r,n_{r+1}+1)) + (1-\lambda_3\delta t)\Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1})) \\
& +(\lambda_4\delta t(p(\lambda_3) + \Pi_{k,t+1}(n-3,n_1,\ldots,n_r,n_{r+1}+2)) + (1-\lambda_4\delta t)\Pi_{k,t+1}(n-2,n_1,\ldots,n_r,n_{r+1}+1)) \\
= \; & \lambda_1\delta t(p(\lambda_1) - \Delta\Pi_{k,t+1}(n,n_1,\ldots,n_r,n_{r+1})) + \Pi_{k,t+1}(n,n_1,\ldots,n_r,n_{r+1}) \\
& -\lambda_2\delta t(p(\lambda_2) - \Delta\Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)) - \Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1) \\
& -\lambda_3\delta t(p(\lambda_3) - \Delta\Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1})) - \Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
& +\lambda_4\delta t(p(\lambda_4) - \Delta\Pi_{k,t+1}(n-2,n_1,\ldots,n_r,n_{r+1}+1)) + \Pi_{k,t+1}(n-2,n_1,\ldots,n_r,n_{r+1}+1) \\
\leq \; & \lambda_1\delta t(p(\lambda_1) - \Delta\Pi_{k,t+1}(n,n_1,\ldots,n_r,n_{r+1})) + \Pi_{k,t+1}(n,n_1,\ldots,n_r,n_{r+1}) \\
& -\lambda_4\delta t(p(\lambda_4) - \Delta\Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)) - \Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1) \\
& -\lambda_1\delta t(p(\lambda_1) - \Delta\Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1})) - \Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
& +\lambda_4\delta t(p(\lambda_4) - \Delta\Pi_{k,t+1}(n-2,n_1,\ldots,n_r,n_{r+1}+1)) + \Pi_{k,t+1}(n-2,n_1,\ldots,n_r,n_{r+1}+1) \\
= \; & -\lambda_1\delta t(\Delta\Pi_{k,t+1}(n,n_1,\ldots,n_r,n_{r+1}) - \Delta\Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1})) \\
& +\lambda_4\delta t(\Delta\Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1) - \Delta\Pi_{k,t+1}(n-2,n_1,\ldots,n_r,n_{r+1}+1)) \\
& +(\Delta\Pi_{k,t+1}(n,n_1,\ldots,n_r,n_{r+1}) - \Delta\Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
= \; & (1-\lambda_1\delta t)(\Delta\Pi_{k,t+1}(n,n_1,\ldots,n_r,n_{r+1}) - \Delta\Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1})) \\
& +\lambda_4\delta t(\Delta\Pi_{k,t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1) - \Delta\Pi_{k,t+1}(n-2,n_1,\ldots,n_r,n_{r+1}+1))
\end{aligned}
$$

This equation means the dynamic process work in one day. So what we need is

$\Delta\Pi_{K-(r+1),T+1}(n,n_1,\ldots,n_r,n_{r+1}) - \Delta\Pi_{K-(r+1),T+1}(n-1,n_1,\ldots,n_r,n_{r+1}) \leq 0$. Which means we need $\Delta_n\Pi_{K-(r+1),T+1}(n,n_1,\ldots,n_r,n_{r+1}) - \Delta_n\Pi_{K(r+1),T+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1) = \Delta_n\Pi_{K-r,1}(n+n_1,n_2,\ldots,n_{r+1},0) - \Delta_n\Pi_{K_r,1}(n+n_1-1,n_2,\ldots,n_{r+1}+1,0) \leq 0$.

After that we will give the proof of $\Delta_n\Pi_{K-r,1}(n+n_1,n_2,\ldots,n_{r+1},0) - \Delta_n\Pi_{K-r,1}(n+n_1-1,n_2,\ldots,n_{r+1}+1,0) \leq 0$,

$$\begin{aligned}
& \Delta_n\Pi_{K-r,1}(n+n_1,n_2,\ldots,n_{r+1},0) - \Delta_n\Pi_{K-r,1}(n+n_1-1,n_2,\ldots,n_{r+1}+1,0) \\
= \; & \Delta\Pi_{K-r,1}(n+n_1,n_2,\ldots,n_{r+1},0) - \Delta\Pi_{K-r,1}(n+n_1-1,n_2,\ldots,n_{r+1}+1,0) \\
= \; & \Pi_{K-r,1}(n+n_1,n_2,\ldots,n_{r+1},0) - \Pi_{K-r,1}(n+n_1-1,n_2,\ldots,n_{r+1},1) \\
& -(\Pi_{K-r,1}(n+n_1-1,n_2,\ldots,n_{r+1}+1,0) - \Pi_{K-r,1}(n+n_1-2,n_2,\ldots,n_{r+1}+1,1)) \\
= \; & \lambda_1\delta t(p(\lambda_1) - \Delta\Pi_{K-r,2}(n+n_1,n_2,\ldots,n_{r+1},0)) + \Pi_{K-r,2}(n+n_1,n_2,\ldots,n_{r+1},0) \\
& -\lambda_2\delta t(p(\lambda_2) - \Delta\Pi_{K-r,2}(n+n_1-1,n_2,\ldots,n_{r+1},1)) - \Pi_{K-r,2}(n+n_1-1,n_2,\ldots,n_{r+1},1) \\
& -\lambda_3\delta t(p(\lambda_3) - \Delta\Pi_{K-r,2}(n+n_1-1,n_2,\ldots,n_{r+1}+1,0)) - \Pi_{K-r,2}(n+n_1-1,n_2,\ldots,n_{r+1}+1,0) \\
& +\lambda_4\delta t(p(\lambda_4) - \Delta\Pi_{K-r,2}(n+n_1-2,n_2,\ldots,n_{r+1}+1,1)) + \Pi_{K-r,2}(n+n_1-2,n_2,\ldots,n_{r+1}+1,1) \\
\leq \; & -\lambda_1\delta t(\Delta\Pi_{K-r,2}(n+n_1,n_2,\ldots,n_{r+1},0) - \Delta\Pi_{K-r,2}(n+n_1-1,n_2,\ldots,n_{r+1}+1,0)) \\
& +\lambda_4\delta t(\Delta\Pi_{K-r,2}(n+n_1-1,n_2,\ldots,n_{r+1},1) - \Delta\Pi_{K-r,2}(n+n_1-2,n_2,\ldots,n_{r+1}+1,1)) \\
& +(\Delta\Pi_{K-r,2}(n+n_1,n_2,\ldots,n_{r+1},0) - \Delta\Pi_{K-r,2}(n+n_1-1,n_2,\ldots,n_{r+1}+1,0)) \\
= \; & (1-\lambda_1\delta t)(\Delta\Pi_{K-r,2}(n+n_1,n_2,\ldots,n_{r+1},0) - \Delta\Pi_{K-r,2}(n+n_1-1,n_2,\ldots,n_{r+1}+1,0)) \\
& +(\Delta\Pi_{K-r,2}(n+n_1,n_2,\ldots,n_{r+1},0) - \Delta\Pi_{K-r,2}(n+n_1-1,n_2,\ldots,n_{r+1}+1,0))
\end{aligned}$$

This means $\Delta\Pi_{K-r,1}(n+n_1,n_2,\ldots,n_{r+1},0) - \Delta\Pi_{K-r,1}(n+n_1-1,n_2,\ldots,n_{r+1}+1,0) \leq 0$ has the transition property in one day, and $\Delta_n\Pi_{K-r,1}(n+n_1,n_2,\ldots,n_{r+1},0) - \Delta_n\Pi_{K-r,1}(n+n_1-1,n_2,\ldots,n_{r+1}+1,0)$ has the transition property cross the boundary. This is the reason while we can get K1.

For K2, $\Delta_{n_{r+1}}\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) - \Delta_{n_{r+1}}\Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+$

$1) \leq 0$, we will show another transition property:

$$\begin{aligned}
& \Delta_{n_{r+1}}\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) - \Delta_{n_{r+1}}\Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+1) \\
= \; & \Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) - \Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
& -(\Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+1) - \Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1})) \\
= \; & \Delta_n\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) - \Delta_n\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1})
\end{aligned}$$

Because

$$\begin{aligned}
& \Delta_n\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) \\
= \; & \Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) - \Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+1) \\
= \; & \Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) - \Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+2) \\
& +\Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+2) - \Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+1) \\
= \; & \Delta\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) + \Delta_{n_{r+1}}\Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+2)
\end{aligned}$$

Then we have

$$
\begin{aligned}
& \Delta_{n_{r+1}}\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1)-\Delta_{n_{r+1}}\Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+1) \\
= \;& \Delta_n\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1)-\Delta_n\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
= \;& \Delta\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1)+\Delta_{n_{r+1}}\Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+2) \\
& -(\Delta\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1})+\Delta_{n_{r+1}}\Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+1)) \\
& \Delta\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1)-\Delta\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}) \\
= \;& \Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1)-\Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+2) \\
& -(\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1})-\Pi_{K-(r+1),t}(n-2,n_1,\ldots,n_r,n_{r+1}+1))
\end{aligned}
$$

$$
\begin{aligned}
& \Delta_{n_{r+1}}\Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}+1)-\Delta_{n_{r+1}}\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1) \\
= \; & \Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}+1)-\Pi_{K-(r+1),t}(n,n_1,\ldots,n_r,n_{r+1}) \\
& -(\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1}+1)-\Pi_{K-(r+1),t}(n-1,n_1,\ldots,n_r,n_{r+1})) \\
= \; & (\lambda_1\delta t(p(\lambda_1)+\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+2)) \\
& +(1-\lambda_1\delta t)\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1}+1)) \\
& -(\lambda_2\delta t(p(\lambda_2)+\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)) \\
& +(1-\lambda_2\delta t)\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1})) \\
& -(\lambda_3\delta t(p(\lambda_3)+\Pi_{K-(r+1),t+1}(n-2,n_1,\ldots,n_r,n_{r+1}+2)) \\
& +(1-\lambda_3\delta t)\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)) \\
& +(\lambda_4\delta t(p(\lambda_4)+\Pi_{K-(r+1),t+1}(n-2,n_1,\ldots,n_r,n_{r+1}+1)) \\
& +(1-\lambda_4\delta t)\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1})) \\
\leq \; & (\lambda_1\delta t(p(\lambda_1)+\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+2)) \\
& +(1-\lambda_1\delta t)\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1}+1)) \\
& -(\lambda_1\delta t(p(\lambda_1)+\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)) \\
& +(1-\lambda_1\delta t)\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1})) \\
& -(\lambda_4\delta t(p(\lambda_4)+\Pi_{K-(r+1),t+1}(n-2,n_1,\ldots,n_r,n_{r+1}+2)) \\
& +(1-\lambda_4\delta t)\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)) \\
& +(\lambda_4\delta t(p(\lambda_4)+\Pi_{K-(r+1),t+1}(n-2,n_1,\ldots,n_r,n_{r+1}+1)) \\
& +(1-\lambda_4\delta t)\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1})) \\
= \; & \lambda_1\delta t(\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+2)-\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)) \\
& +(1-\lambda_1\delta t)(\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1}+1)-\Pi_{K-(r+1),t+1}(n,n_1,\ldots,n_r,n_{r+1})) \\
& -\lambda_4\delta t(\Pi_{K-(r+1),t+1}(n-2,n_1,\ldots,n_r,n_{r+1}+2)-\Pi_{K-(r+1),t+1}(n-2,n_1,\ldots,n_r,n_{r+1}+1)) \\
& -(1-\lambda_4\delta t)(\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}+1)-\Pi_{K-(r+1),t+1}(n-1,n_1,\ldots,n_r,n_{r+1}))
\end{aligned}
$$

Appendix .C Codes of the Programme

```
function v=rental_duration(a1,b1,X,K,T) %a,b
%D=zero(m,n);
%D(m,n)=zeros(K,T,X,X,X);
D=zeros(K,T,X,X,X,X); %optimal expect revenuegiven k and t, x=1
DP=zeros(K,T,X,X,X,X); %optimal pricegiven k and t, x=1
for k=K:-1:1
    if k>1
      for t=T-1:-1:1
            for x1=1:1:X
                  for x2=X-x1+1:-1:1
                    for x3=X-x1-x2+2:-1:1
                      if x1>1
                       D(k,t,x1,x2,x3,X-x1-x2-x3+3)=maxr1(a1,b1,D
                           (k,t+1,x1-1,x2,x3,X-x1-x2-x3+4),D(k,t
                           +1,x1,x2,x3,X-x1-x2-x3+3));
                       DP(k,t,x1,x2,x3,X-x1-x2-x3+3)=optprice(a1,
                           b1,D(k,t+1,x1-1,x2,x3,X-x1-x2-x3+4),D(k
                           ,t+1,x1,x2,x3,X-x1-x2-x3+3));
                      else
                       D(k,t,x1,x2,x3,X-x1-x2-x3+3)=D(k,t+1,x1,x2
                           ,x3,X-x1-x2-x3+3);
                       DP(k,t,x1,x2,x3,X-x1-x2-x3+3)=a1/b1;
                      end
                      if t==1 && X-x1-x2-x3+3==1
                       for x4=1:1:x1
                         D(k-1,T,x4,X-x2-x3-x4+3,x2,x3)=D(k,t,x1,
```

```
                        x2 , x3 ,X–x1−x2−x3+3);
                    end
                  else
                  end
                end
              end
          end
      end
else
  for t=T−1:−1:1
      for x1=1:1:X
          for x2=X–x1+1:−1:1
            for x3=X–x1−x2+2:−1:1
              if x1>1
                D(k,t,x1,x2,x3,X–x1−x2−x3+3)=maxr1(a1,b1,D
                    (k,t+1,x1−1,x2,x3,X–x1−x2−x3+4),D(k,t
                    +1,x1,x2,x3,X–x1−x2−x3+3));
                DP(k,t,x1,x2,x3,X–x1−x2−x3+3)=optprice(a1,
                    b1,D(k,t+1,x1−1,x2,x3,X–x1−x2−x3+4),D(k
                    ,t+1,x1,x2,x3,X–x1−x2−x3+3));
              else
                D(k,t,x1,x2,x3,X–x1−x2−x3+3)=D(k,t+1,x1,x2
                    ,x3,X–x1−x2−x3+3);
                DP(k,t,x1,x2,x3,X–x1−x2−x3+3)=a1/b1;
              end
            end
          end
      end
  end
```

```
        end
end
%v=D;
v=DP(1,1,X,1,1,1);
%v=DP;

a0=20;
b0=10;
x0=12;
k0=8;
t0=21;

P=rental_duration(a0,b0,x0,k0,t0);
average_price=zeros(1000);
average_inventory=zeros(1000);
Number_customer=zeros(1000);
averageprice=0;
averageinventory=0;
total_customer=0;
for j=1:1000
state=[1 1 x0 1 1 1];
state_compare=[0 0 0 0 0 0];
price=zeros(k0*(t0-1));
probability=zeros(k0*(t0-1));
total_price=0;
total_inventory=0;
        for i=1:k0*(t0-1)
        probability(i)=f_2(state,P);
        state_compare=transfer_2(state, f_2(state,P),k0);
```

```
        if state_compare(3)<state(3)
            Number_customer(j)= Number_customer(j)+1;
        else
        end
       state=state_compare;
       price(i)=(a0-probability(i)*20)/b0;
       total_price=total_price+price(i);
       total_inventory=state(3)+total_inventory;
      end
 average_price(j)=total_price/i;
 average_inventory(j)=total_inventory/i;
%fprintf('%f\n', average_price(j)) ;
end
for jj=1:1000
     %averageinventory=averageinventory+average_inventory(jj);
    averageprice=averageprice+average_price(jj);
    total_customer=total_customer+Number_customer(jj);
end
 averageprice=averageprice/1000;
 averagecustomer=total_customer/1000;
%for t=1:200
%total_average_price=0;
%total_average_price=total_average_price+average_price(t);
%end
%average_price=total_average_price/t;
%determin_price(t)=0.72;
 t=1:200;
 plot(t,average_price(t));
%plot(t,average_price(t),t,determin_price(t));
```

```
function final_state=transfer_2(current_state , prob,k_n)
    r = unifrnd(0,1);
    if r < prob && current_state(3)>1
        next_state=[current_state(1), current_state(2)+1,
            current_state(3)-1, current_state(4), current_state
            (5),current_state(6)+1];
        if next_state(2)>20 && next_state(1)<k_n
            next_state = [next_state(1)+1, next_state(2)-20,
                next_state(3)+next_state(4)-1, next_state(5),
                next_state(6), 1];
        end
    else
        next_state=[current_state(1), current_state(2)+1,
            current_state(3), current_state(4), current_state(5),
             current_state(6)];
        if next_state(2)>20 && next_state(1)<k_n
            next_state=[next_state(1)+1, next_state(2)-20,
                next_state(3)+next_state(4)-1,next_state(5) ,
                next_state(6) ,1];
        end
    end
    final_state=next_state;
```