Sharing Work Dynamically on U-Lines: System Productivity and Individual Remuneration

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by
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Abstract

Inspired by the concept of cellular bucket brigades, we propose simple rules for workers to share work on U-shaped lines with discrete work stations. For a three-station U-line with a worker-specific velocity setting, we identify the policies that maximize system productivity and the policies that maximize each worker's remuneration rate. For a team with a faster worker and a slower worker, we find that the faster worker's preferred policies maximize system productivity for most work-content distributions. Whenthe policies preferred by the system, the faster worker, and the slower worker are all different, we find a way to resolve the tripartite conflict. On the other hand, if both workers prefer the same policy then this policy also maximizes system productivity. For an M-station U-line with a worker- and station-specific velocity setting, we show that the system always converges to a fixed point or a period-2 orbit. We provide a sufficient condition for the fixed point to be a global attractor. We also develop algorithms to determine the fixed point and the corresponding throughput. We find that increasing the number of stations generally improves throughput for certain work-content distributions. However, further dividing the U-line into more stations has diminishing returns.

Contents

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Chapter 1

Introduction

Allocating workload to workers in an assembly line where there are more work stations than workers can be challenging as it requires effective coordination of workers so that their idle time is minimized and the system's throughput is maximized. One way to coordinate workers along an assembly line is to organize them as a bucket brigade (Bartholdi and Eisenstein 1996a). When workers form a bucket brigade on an assembly line, each worker assembles his item (an instance of the product) until it is taken over by a downstream colleague or he completes his item if he is the last worker of the line. After that the worker walks back to take over an item from an upstream colleague or to initiate a newitem at the start of the line if he is the first worker.

Under certain assumptions, Bartholdi and Eisenstein (1996a) showthat if workers are sequenced from slowest to fastest according to their work velocities in the direction of production flow, then a bucket brigade will *self-balance* such that the hand-off locations between any two neighboring workers will converge to a fixed point and every worker repeatedly works on a fixed portion of the line. Furthermore, the system's throughput attains a level that is the maximum possible for the system.

The most widely known application of bucket brigades is order-picking in distribution centers (Bartholdi and Eisenstein 1996b and Bartholdi et al. 2001). In a large distribution center, fast-moving products are often stored on racks along a single aisle so that workers can quickly pick the products to fulfill customer orders. Bucket brigades are especially effective in this setting for the following reasons: (1) The rule is simple for workers to learn and follow. (2) Due to their self-balancing property, we need neither a work-content model nor computation for work balance, which are required by any static work-allocation strategy. (3) Since workers dynamically and constantly balance their work, the system can restore balance from temporary disruptions and is adaptive to changes in products' demand seasonality.

Bucket brigades are also used in the production of garments, packaging of cellular phones, and assembly of tractors, large-screen televisions, and automotive electrical harnesses (Bartholdi and Eisenstein 1996a, b, Bartholdi and Eisenstein 2005, and Villalobos et al. 1999a, b).

Lim (2011) introduces the ideas of *cellular* bucket brigades to reduce unproductive travel of workers. Under his new design, the work content is distributed on both sides of an aisle so that each worker works on one side when he proceeds in one direction and works on the other side when he proceeds in the reverse direction.

Since workers work in both directions, the throughput can be improved significantly.

Up to now, the ideas of bucket brigades are extensively studied and are successfully applied on assembly lines with a straight-line layout. In this chapter we introduce newrules, inspired by the concept of cellular bucket brigades, for workers to share work on U-shaped assembly lines. Specifically, we study a three-station U-line with two workers. We assume at any point in time no more than one worker can work on any station (for example, due to limited tools or equipment). A worker is *blocked* when he is about to enter a station but his co-worker is still working on the station. This causes unproductive idle time. It is non-trivial to coordinate the workers so that they can share their work efficiently.

U-lines are commonly used in practice because they possess several advantages over straight lines. These include providing better visibility and communications and thus, leading to better quality control (Miltenburg and Wijingaard 1994). Furthermore, many firms adopt a U-shaped layout due to space constraints.

In this paper, we adopt the ideas of cellular bucket brigades by Lim (2011) and propose rules for workers to share work on the U-line. We believe our work is the first to analytically address dynamic work-sharing on U-lines with discrete work stations.

After review the related literature, we first introduce cellular bucket brigade rules for two workers to share work on a three-station U-line with discrete work stations in Chapter 3. We fully analyze the system where each worker maintain the same velocity over all stations. We identify the policies that maximize the system's throughput and the policies that maximize individual workers' remuneration rates. We then generalize the three-station U-line to M-station U-line where each worker may have different workvelocities in different stations in Chapter 4. We fully analyze the three-station case with this general work velocity setting and provide efficient algorithms to compute the fixed point and throughput of the M-station case. Finally, we conclude our work in Chapter 5.

Chapter 2

RelatedLiterature

Bartholdi and Eisenstein (1996a) introduce bucket brigades as a way to coordinate workers along an assembly line with more stations than workers. When workers form a bucket brigade on an assembly line, each worker assembles his item until it is taken over by a downstream colleague or he completes his item if he is the last worker of the line. After that the worker walks back to take over an item from an upstream colleague or to initiate a newitem at the start of the line if he is the first worker.

Bartholdi and Eisenstein (1996a) study a model with deterministic work content. Each worker has a deterministic, finite work velocity and an infinite walk-back velocity. They showthat if workers are sequenced from slowest to fastest according to their work velocities in the direction of production flow, then the system will *self-balance*: The hand-off locations between any two neighboring workers will converge to a *fixed point* and every worker repeatedly works on a fixed portion of the line.

Furthermore, the long-run average throughput will achieve the maximum possible for the system if the work content is continuously and uniformly distributed.

The most widely known application of bucket brigades is order-picking in distribution centers (Bartholdi and Eisenstein 1996b and Bartholdi et al. 2001). Bucket brigades are also used in the production of garments, packaging of cellular phones, and assembly of tractors, large-screen televisions, and automotive electrical harnesses (Bartholdi and Eisenstein 1996a, b, Bartholdi and Eisenstein 2005, and Villalobos et al. 1999a, b).

Based on the same model, Bartholdi et al. (1999) study the dynamics of two- and three-worker bucket brigades with workers not necessarily sequenced from slowest to fastest. Bartholdi et al. (2001) consider stochastic work content on work stations. They find that the dynamics and throughput of the stochastic system will be similar to that of the system with deterministic work content when there is sufficient work distributed among sufficiently many stations. They also describe the effectiveness of bucket brigades in order-picking in a distribution center, which experienced a 34% increase in productivity after the workers began picking orders by bucket brigades.

Bartholdi and Eisenstein (2005) extend the basic model of bucket brigades to capture walk-back time and hand-off time. Bartholdi et al. (2009) consider the case where workers are allowed to overtake or pass each other and they walk back with finite velocities. The authors show that the system may exhibit chaotic behavior that causes the inter-completion times of items to be effectively random, even though the model is purely deterministic. The system can avoid such pathologies if workers are indexed from most impeded by work to least impeded by work.

Armbruster and Gel (2006) assume workers' work velocities do not dominate each other along the entire line. They study the dynamics and throughput of a two-worker system. Armbruster et al. (2007) consider a model where workers improve their work velocities as they learn. Webster et al. (2011) examine the performance of a bucket brigade order-picking system by changing the distribution of products along anaisle. They identify conditions where product distribution has a large impact on throughput.

Lim and Yang (2009) analyze the dynamics of bucket brigades on discrete work stations and identify the best policies that maximize the system's throughput. They show that the policy that fully cross-trains the workers and sequences them from slowest to fastest is not always the best for the system, even though it outperforms other policies for most work-content distributions. Gurumoorthy et al. (2009) study an M-station, two-worker bucket brigade. They assume each worker has different work velocities on different stations. They determine the asymptotic dynamic behavior and the throughput of the system using an algorithmic approach.

Kirkizlar et al. (2011) study dynamic assignment of workers to stations in tandem lines with more stations than workers. They consider buffers between stations. The authors findflexibilitystructures and worker assignment policies that maximize the system's throughput. For an excellent review of workforce cross-training and coordination, see Hopp and Van Oyen (2004).

Eisenstein(2005) studies a productionsystem where facilities follow a cyclic schedule to replenish their inventory by a shared resource. Using the ideas of self-balance, he proposes a dynamic produce-up-to policy to recover a target schedule after disruptions.

Bischak (1996) considers a U-shaped manufacturing module with fewer workers than stations. She proposes rules, which are effectively for a straight-line layout, for workers to move in the module. The throughput and flow time of this moving-worker module are compared with a system with one dedicated worker per station through simulation studies. Chand and Zeng (2001) consider static work allocation and compare U-lines with straight-line layouts under the impact of stochastic task times.

Lim (2011) presents a design alternative of bucket brigades that may provide significant improvement in throughput. Under the new design, each worker works on one side of an aisle when he proceeds in one direction and works on the other side when he proceeds in the reverse direction. The author proposes the cellular bucket brigade rules for workers to share work under the new design. He also finds a sufficient condition for the system to self-balance. Numerical examples suggest that the system under the new design can be 30% more productive than a traditional bucket brigade.

Chapter 3

System productivity and individual remunerationona three-stationU-line

3.1 Introduction

U-lines with three stations and two workers are common not only in manufacturing, but also in the service industry. For example, a worker at the counter of a cafeteria first takes an order from a customer. He does some preparation work for the order before he passes it to a co-worker, who is usually stationed in the kitchen to prepare food. When the food is ready, it is taken over by the first worker at the counter, who puts the complete order on a tray and serves the customer. Another example can be seen in a pharmacy, where a pharmacist at the counter first handles a prescription and identifies the drugs needed. He then asks a colleague to pick up the drugs from a store room. After the drugs are picked up, they are taken over by the pharmacist, who double-checks the drugs and prints out a receipt before he passes them to the customer.

The goal of this chapter is to provide answers to the following questions:

1. Howshould we coordinate the workers on U-lines with discrete work stations so that they can share work efficiently? We introduce simple rules for workers to share work on a U-line withdiscrete work stations. By following these rules, workers work in both directions and so they can dynamically balance their workload without too much unproductive travel.

2. To what extent should we cross-train the workers? For the system to be more productive, is it always necessary to train workers to work on all stations? We first assume workers are *fully cross-trained* so that each of them can work on all stations of a U-line. We analyze the asymptotic dynamic behavior of the system and determine its long-run average throughput. We then compare the fully cross-trained team with a *partially cross-trained* team in which each worker is only trained to work on some of the stations. We find that under some situations the partially cross-trained team is more productive than the fully cross-trained team.

3. What are the policies that maximize system productivity? Givena work-content distribution, we find these policies by examining the performance of

different orderings of workers (according to their work velocities) combined with different extents of cross-training.

4. How shouldwe pay the workers? What are the policies preferred bythe workers? When workers share their work on a U-line or a bucket brigade, they usually do not contribute equally to the system's throughput. To reward workers according to their contribution, we introduce a *remuneration rate* for each worker. Workers are paid according to their share of work and the system's throughput. We identify the policies that maximize each worker's remuneration rate.

5. Are the policies that maximize system productivity always consistent with the policies preferred by the workers? If not, how should we resolve the conflict?We find that individual workers' preferences are not always consistent with the system's preference. Between the faster and the slower workers, the system's preference is consistent with the faster worker's preference for more work-content distributions. Furthermore, if workers have very different work velocities, following the preference of the faster worker almost always maximizes the system's throughput. We also suggest a way to resolve conflict when the policies preferred by the system, the faster worker, and the slower worker are all different.

In this chapter, we first introduce cellular bucket brigade rules for workers to share work on U-lines with discrete work stations. We then analyze the dynamics and determine the throughput of a fully cross-trained team, and compare the fully cross-trainedteam with a partially cross-trained team. We identify the policies that maximize the system's throughput and the policies that maximize individual workers' remuneration rates. We also introduce a simple heuristic to maximize the system's throughput. Finally, we study the conflict and consistency between the system's and the workers' preferences before we conclude our work.

3.2 Cellular bucket brigades on U-lines

Consider a U-shaped assembly line with three work stations shown in Figure 3.1(a). Stations 1 and 3 are separated by an aisle while station 2 spans across the aisle. Each item is initiated at the start of station 1 and is progressively assembled on the same sequence of stations until it is completed at the end of station 3. Let s_j denote the work content of station j, for $j = 1, 2, 3$. We normalize the total work content of the line such that $\sum_{j=1}^{3} s_j = 1$.

The assembly line can be conceptualized as a line segment with length 1. Figure 3.1(b) shows such a conceptual line, which is represented by a bold solid line. Locations 0 and 1 represent the start and the end, respectively, of the line. The intervals $[0, s_1]$, $(s_1, s_1 + s_2]$, and $(s_1 + s_2, 1]$ on the line represent the work content of stations 1, 2, and 3 respectively.The horizontal line segments $[0 \quad, s_1]$ and $(s_1 + s_2, 1]$ are parallel to each other, and the line segment $(s_1, s_1 + s_2]$ is perpendicular to them.

We consider a team of two workers. Worker i is cross-trained to work on *zone*

Z_i — a set of stations on the line, for $i = 1, 2$. Each worker assembles a single item from station to station in his zone and only one worker is allowed to work on a station at a time. We assume worker i works with velocity v_i on all stations in his zone Z_i and the time to walk from one station to another is negligible. We first consider a fully cross-trained team such that $Z_1 = Z_2 = \{1, 2, 3\}$.

Define x_i as the *horizontal position* of worker i. This horizontal positionis determined by projecting the *actual location* of the worker on the conceptual line to the horizontal axis. Figure 3.1(b) shows the relationship between the actual location and the horizontal position of each worker.To distinguish these two coordinate systems, we call any point on the conceptual line a *location* and any point on the horizontal axis a *position.*

We set the origin of the horizontal axis to be the projection of location 0 (the start of station 1) to the axis. Note that a horizontal position can be negative if $s_1 < s_3$. Since station 2 runs vertically across the aisle, we have $x_i \leq s_1$, for $i = 1, 2$. We require the workers to remain in a fixed sequence along the horizontal axis such that $x_1 \leq x_2$ at any point in time.

We say worker 1, who is working on station 1, *meets* worker 2, who is working on station 3, when their horizontal positions coincide (that is, $x_1 = x_2$). When worker 1 meets worker 2, a *hand-off* between the two workers occurs: Each worker first relinquishes his item, walks across the aisle, and takes over each other's item. After the hand-off, worker 1 works on station 3 while worker 2 proceeds on station 1.

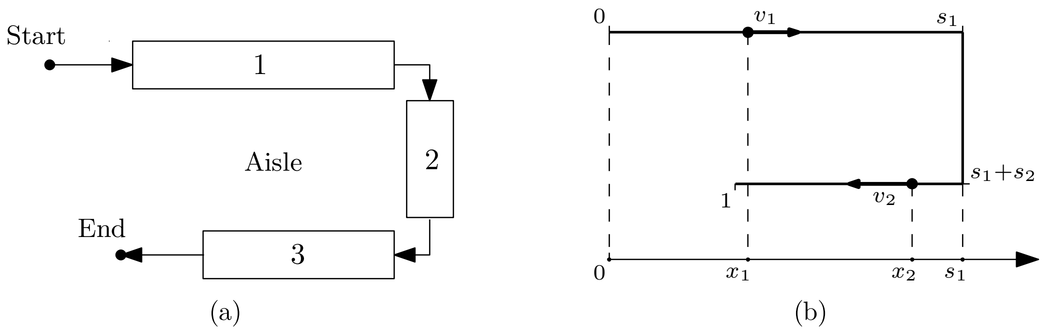


Figure 3.1: **A U-shaped assembly line.** (a) Each item is initiated at the start of station 1 and is progressively assembled on the same sequence of stations until it is completed at the end of station 3. (b) The assembly line is conceptualized as a line segment with length 1. The horizontal position x_i is determined by projecting the location of worker i on the conceptual line to the horizontal axis.

Aworker is blocked if he reaches the start of a station while his colleague is still working on the station. In Figure 3.1(b), worker 1 can be blocked only at locations 0 and s_1 and worker 2 can be blocked only at location $s_1 + s_2$. Note that if worker 1 is blocked at location s_1, then a hand-off occurs immediately after worker 2 finishes his work on station 2. After the hand-off, worker 1 works on station 3 while worker 2 reenters station 2. As a result, worker 1 can never work on station 2.

Worker 2 is *halted* if he reaches the end of station 3 (location 1) before he meets worker 1. Halting is possible only if $s_1 > s_3$. If worker 2 is halted, he remains idle until a hand-off occurs when the horizontal positions of the two workers coincide.

Figure 3.2 shows how a cellular bucket brigade operates on the U-line. Let x^k denote the k-th hand-off position. At the k-th hand-off, the two workers first relinquish their work andthen walk across the aisle. After they exchange their work, worker 1 works on station 3 with velocity v_1. When he finishes his work on station 3, he walks instantaneously to the start of station 1, initiates a new item,

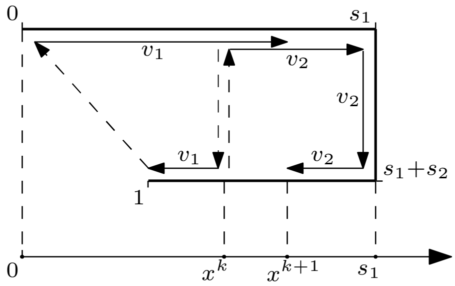


Figure 3.2: **A cellular bucket brigade on the U-line.** This figure shows the paths of the two workers between the k-th and $(k+1)$-st hand-offs. The solid arrows correspond to working, while the dashed arrows correspond to instantaneous walk.

and works on station 1. Meanwhile, worker 2 works on station 1 with velocity v_2. After he reaches the end of station 1, he continues to work on stations 2 and 3 until he meets worker 1 again at position x^{k+1}.

Specifically, each worker in the system follows the simple rules below:

Work onstation 1: Continue to assemble your item until

1. you exchange work with your colleague if you are worker 1, then **work onstation 3** ; or
2. youfinishthe workonstation1 if youare worker 2, then **work on station2** .

Work onstation 2: Continue to assemble your item until you finish the work on station 2, then **work onstation3** .

Work onstation 3: Continue to assemble your item until

1. you finish the work on station 3 if you are worker 1, then initiate a new item and **work onstation1** ; or
2. you exchange work with your colleague if you are worker 2, then **work onstation 1** .

Note that these rules can be easily general to U-lines with more stations and workers.

3.3 Dynamics andthroughput

According to the cellular bucket brigade rules, if $s_1 > s_3$ then $x_1 \in [0, s_1]$ and $x_2 \in [s_1 - s_3, s_1]$. Otherwise, $x_1 \in [s_1 - s_3, s_1]$ and $x_2 \in [0, s_1]$. Note that any hand-off position falls in the interval $I = [\max(s_1 - s_3, 0), s_1]$ on the horizontal axis. Let $f : I \mapsto I$ be a function, defined implicitly by the cellular bucket brigade rules, such that $x^{k+1} = f(x^k)$. The sequence of iterates $x^1, x^2, x^3, \ldots$ is called the *orbit* of an initial iterate x^0 under f. We say x^* is a fixed point if $x^* = f(x^*)$. A *period-2 orbit* is an orbit that alternates between p and q, where $p = f(q)$ and $q = f(p)$ (see Alligood et al. (1996)). Note that $f(f(p)) = p$ and $f(f(q)) = q$.

Let $r = v_1/v_2$ denote the velocity ratio. We first construct the function f and then determine the asymptotic behavior of the cellular bucket brigade by analyzing the function. The details can be found in Appendix A.1.1. A distribution of work content on stations can be uniquely represented by a point (s_1, s_3) in Figure 3.3(a), which shows all possible work-content distributions.For a given velocity ratio r, the work-content distributions in Figure 3.3(a) can be partitioned into five regions. For example, Region 1 contains systems with large s_1 and small s_3. Each region corresponds to a distinct asymptotic behavior. Theorem 1 summarizes the asymptotic behavior and long-run average throughput of a fully cross-trained team in each region.

Theorem 1. *If $Z_1 = Z_2 = \{1, 2, 3\}$, the cellular bucket brigade has five different asymptotic behaviors described as follows.*

Region1 $(s_3 < -r + (r+1)s_1)$**:** *The system converges to a fixed point* $x^* = s_1 - s_3$*. At the fixed point, worker 1 is constantly blocked at location* 0 *and worker 2 is constantly halted at location* 1*. The average throughput is* $\mathcal{T}^F = \frac{v_1}{s_1+(r-1)s_3}$*.*

Region2 $\left(s_3 \geq -r + (r+1)s_1 \text{ and } s_1 > \frac{r}{r+1} - \frac{r-1}{r+1}s_3\right)$**:** *The system converges to a fixed point* $x^* = \frac{r}{r+1} - \frac{r}{r+1}s_3$*. At the fixed point, worker 1 is constantly blocked at location* 0*. The average throughput is* $\mathcal{T}^F = \frac{(r+1)v_2}{(r+1)s_1+(1-r)(1-s_3)}$*.*

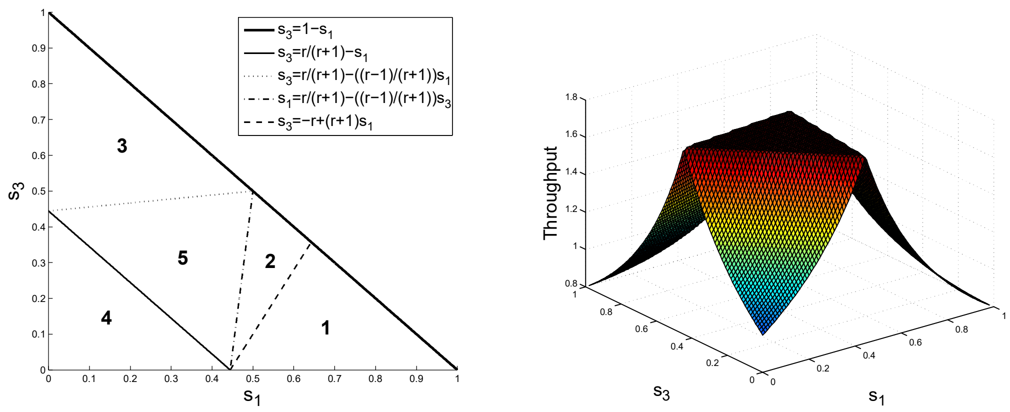
Region3 $\left(s_3 > \frac{r}{r+1} - \frac{r-1}{r+1}s_1\right)$**:** *The system converges to a fixed point* $x^* = \frac{r}{r+1}s_1$*. At the fixed point, worker 2 is constantly blocked at location* $s_1 + s_2$*. The average throughput is* $\mathcal{T}^F = \frac{(r+1)v_1}{(r+1)s_3+(r-1)s_1}$*.*

Region4 $\left(s_3 < \frac{r}{r+1} - s_1\right)$**:** *The system converges toa fixedpoint* $x^* = s_1$*. At the fixed point, worker 1 is constantly blocked at location* s_1*. The average throughput is* $\mathcal{T}^F = \frac{v_2}{1-s_1-s_3}$*.*

Region5 $\left(s_1 \leq \frac{r}{r+1} - \frac{r-1}{r+1}s_3,\ s_3 \leq \frac{r}{r+1} - \frac{r-1}{r+1}s_1 \text{ and } s_3 \geq \frac{r}{r+1} - s_1\right)$**:** *The system converges to a period-2 orbit:* x *and* $\frac{r}{r+1} + s_1 - s_3 - x$*, where* x *depends on the initial locations of the workers. Neither blocking nor halting occurs onthe period-2 orbit. The average throughput is* $\mathcal{T}^F = v_1 + v_2$*, the maximum possible for the system.*

Proof. See Appendix A.1.1 □

Note that sometimes increasing workers' velocities may decrease system productivity. For example, for $v_1 = 1.5, v_2 = 1,\ s_1 = 0.6, s_3 = 0.1$ falls in Region



(a) (b)

Figure 3.3: **Asymptotic behaviors and throughput.** (a) The cellular bucket brigade has different asymptotic behaviors in different regions. (b) The system has the highest long-run average throughput in Region 5. The throughput decreases as the system moves away from Region 5. For both graphs, we set $v_1 = 0.8$ and $v_2 = 1$.

2 and the corresponding throughput is $\mathcal{T}^{\mathrm{F}} = 2.3810$. If we increase v_2 to 1.05, $s_1 = 0.6, s_3 = 0.1$ still falls in Region 2 but the corresponding throughput decreases to 2.3800. This is mainly because increasing v_2 leads to longer blocking time for worker 1.

Figure 3.3(b) shows the long-run average throughput of the cellular bucket brigade. The system has the highest throughput in Region 5, where the workers are fully utilized. The throughput decreases as the system moves away from Region 5. At each of the boundaries of Region 5, the period-2 orbit degenerates to a fixed point. Note that only at the boundaries of Region 5 (that is, $s_1 = \frac{r}{r+1} - \frac{r-1}{r+1}s_3$, $s_3 = \frac{r}{r+1} - \frac{r-1}{r+1}s_1$, and $s_3 = \frac{r}{r+1} - s_1$) the system converges to a fixed point with the maximum possible throughput $v_1 + v_2$.

3.4 Full versus partial cross-training

We compare the performance of the fully cross-trained team with a partially cross-trained team with $Z_1 = \{1, 2, 3\}$ and $Z_2 = \{2\}$. In the partially cross-trained team worker 2 only works on station 2 and follows the rule below.

Work onstation 2: Continue to assemble your item until you finish the work on station 2 and exchange work with your colleague, then **work onstation2** .

On the other hand, worker 1 follows the rules below.

Work onstation 1: Continue to assemble your item until you exchange work with your colleague, then **work onstation3** .

Work onstation 3: Continue to assemble your item until you finish the work on station 3, then initiate a new item and **work onstation 1** .

Since the two workers always exchange work at horizontal position s_1 inthe partially cross-trained team, we have the following theorem.

Theorem 2. *If $Z_1 = \{1, 2, 3\}$ and $Z_2 = \{2\}$, the system always operates on a fixed point $x^* = s_1$ and has two different asymptotic behaviors in different regions of Figure 3.3(a):*

Regions 1, 2, 3, and 5 $\left(s_3 \geq \frac{r}{r+1} - s_1\right)$**:** *Worker 1 is never idle and the average throughput is $\mathcal{T}^P = \frac{v_1}{s_1+s_3}$.*

Region4 $\left(s_3 < \frac{r}{r+1} - s_1\right)$**:** *Worker 2 is never idle and the average throughput is $\mathcal{T}^P = \frac{v_2}{1-s_1-s_3}$.*

Note that both the fully and partially cross-trained teams lead to the same asymptotic behavior and throughput in Region 4. This is because station 2 is the bot-

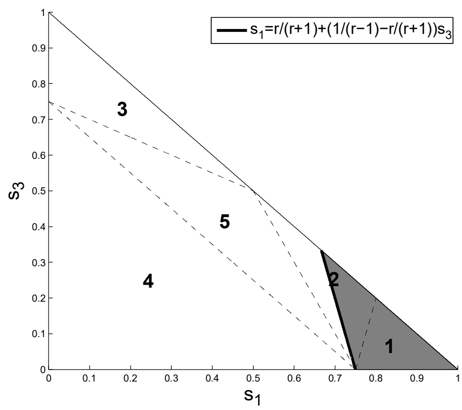


Figure 3.4: **Full versus partial cross-training.** The fully cross-trainedteam dominates in all circumstances except for $r > 2$ and $s_1 > \frac{r}{r+1} + \left(\frac{1}{r-1} - \frac{r}{r+1}\right) s_3$ (the shaded area). In this graph, we set $r = 3$.

tleneck in this region andthe system always converges to the same fixedpoint, independent of how the workers are cross-trained.

We first compare the throughput of the fully andthe partially cross-trained teams. We then examine the preference of each worker between the two teams.

3.4.1 System productivity

Comparing Theorems 1 and 2, we have the following result.

Corollary 1. *The fully cross-trainedteam is at least as productive as the partially cross-trainedteam in all circumstances except for* $r > 2$ *and* $s_1 > \frac{r}{r+1} + \left(\frac{1}{r-1} - \frac{r}{r+1}\right) s_3$ *(the shaded area in Figure 3.4) where the partially cross-trained team is more productive.*

Proof. See Appendix A.1.2 □

Corollary 1 implies that fully cross-training the workers does not necessarily lead to higher system productivity. In the shaded area of Figure 3.4, station 1 has significantly more work content than station 3. If worker 2 is fully cross-trained, he constantly takes over work on station 1 from worker 1, who is then blocked for a significant amount of time at the start of station 1.This waste of production capacity is especially significant if worker 1 is substantially faster than worker 2. Corollary 1 shows that if $r > 2$ (worker 1 is more than two times faster than worker 2), the system is more productive in the shaded area if worker 2 is restricted to work only on station 2. Thus, to maximize the average throughput we should fully cross-train the workers except when $r > 2$ and $s_1 > \frac{r}{r+1} + \left(\frac{1}{r-1} - \frac{r}{r+1}\right) s_3$.

In practice, system productivity is not the only concern of a manager. Another important question in a work-sharing team is whether the workers are satisfied with their rewards given their contribution to the system's output. Since each worker may contribute to different extents under different teams (fully or partially crossed-trained), they have their own preferences on cross-training.

3.4.2 Individual remuneration

Although full cross-training results in higher system productivity in many situations according toCorollary 1, does full cross-training benefit both workers? Define $\alpha_i \in [0, 1]$, for $i = 1, 2$, as the average portion of work content of each item covered by worker i. By definition, α_i depends on the asymptotic behavior of the system (fixed point or period-2 orbit) and $\alpha_1 + \alpha_2 = 1$. We normalize the total income of the team such that the manager pays 1 dollar per item produced to the team. We assume each worker i is paid according to his contribution such that he obtains α_i dollar per item produced. Define the remuneration rate of worker i as his income per unit time, which is equal to α_i times the average throughput of the system. For each region of Figure 3.3(a), the following theorem compares the remuneration rate of each worker in the fully cross-trained team with that in the partially cross-trained team.

Theorem 3. *Different workers have different preferences on cross-training in each region of Figure 3.3(a):*

1. *Worker 1 obtains a higher remuneration rate in the partially cross-trained team in Regions 1 and2, and his remuneration rate is the same for both teams in Regions 3, 4, and 5.*

2. *Worker 2 obtains a higher remuneration rate in the fully cross-trained team in Regions 1, 2, 3, and 5 except at the boundary $s_3 = \frac{r}{r+1} - s_1$, and his remuneration rate is the same for both teams in Region 4 and at the boundary $s_3 = \frac{r}{r+1} - s_1$.*

Proof. See Appendix A.1.3 □

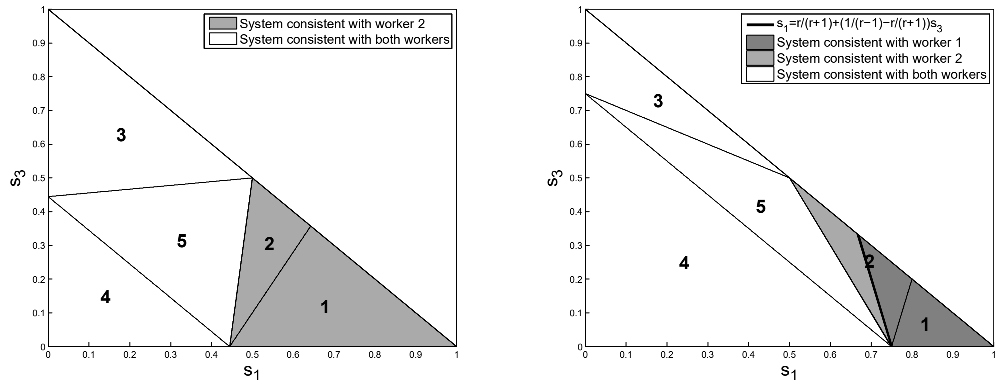
The proof of Theorem 3 shows that eachworker prefers to be busyso that they can earn more. Worker 2 is busier in the fully cross-trained team than in the partially cross-trained team, so full cross-training always benefits worker 2. Worker 1 always prefers partial cross-training, especially in Regions 1 and 2, where s_1 is significantly larger than s_3. This is because worker 1 does not want to be interrupted by worker 2 on station 1 where there is relatively more work content.

Corollary1 shows that the fully cross-trainedteam results inhigher system productivity in many circumstances. However, Theorem 3 implies that the fully cross-trained team is not preferred by worker 1. Thus, the system's preference may contradict individual workers' preferences. We will compare the system's and the individuals' preferences on cross-training under different conditions.

3.4.3 System versus individual preferences

Figures 3.5(a) and (b) compare the system's and individual workers' preferences for $r \leq 2$ and $r > 2$ respectively. For both graphs, the system and workers prefer full cross-training inRegions 3 and 5; and they are indifferent to full or partial cross-training in Region 4. If $r \leq 2$, both the system and worker 2 prefer full cross-training in the shaded area of Figure 3.5(a), contrary to the preference of worker 1. If $r > 2$, both the system and worker 1 prefer partial cross-training in the dark shaded area of Figure 3.5(b), contrary to the preference of worker 2. On the other hand, both the system and worker 2 prefer full cross-training in the light shaded area of Figure 3.5(b), contrary to the preference of worker 1.

Up to now, we assume the ordering of workers along the horizontal axis is fixed (for example, the faster worker is always worker 2) and analyze system productivity and individual remuneration in the fully and partially cross-trained teams. Given two workers, let $v_{\min}$ and $v_{\max}$ denote the velocities of the slower and faster workers



(a) $r = 0.8$ (b) $r = 3$

Figure 3.5: **System versus individual preferences.** (a) For $r \leq 2$, the system and worker 2 prefer full cross-training in the shaded area. (b) For $r > 2$, the system and worker 1 prefer partial cross-training in the dark shaded area; and the system and worker 2 prefer full cross-training in the light shaded area. For both graphs, the system and both workers prefer full cross-training in Regions 3 and 5; and they are indifferent to full or partial cross-training in Region 4.

respectively. Suppose now we have the ability to change the ordering of workers along the horizontal axis. That is, we can let the faster worker be worker 1 or worker 2. What are the effects on system productivity and individual remuneration?We answer this question in the following section.

3.5 Best policies for system and preferred policies for individuals

Given the work velocities of workers, assume the manager can make the following decisions: (1) Choose the ordering of workers along the horizontal axis. (2) Choose either the fully or the partially cross-trained team. A *policy* is a combination of the ordering of workers and the extent of cross-training. Table 3.1 shows four possible policies for the system. Worker 1 is slower than worker 2 if we order the workers from slowest to fastest.

Table 3.1: **Policies.** A policy is a combination of the ordering of workers and the extent of cross-training.

	Fully Cross-Trained	**Partially Cross-Trained**
Slowest to Fastest	SF	SP
Fastest to Slowest	FF	FP

3.5.1 Best policies for system

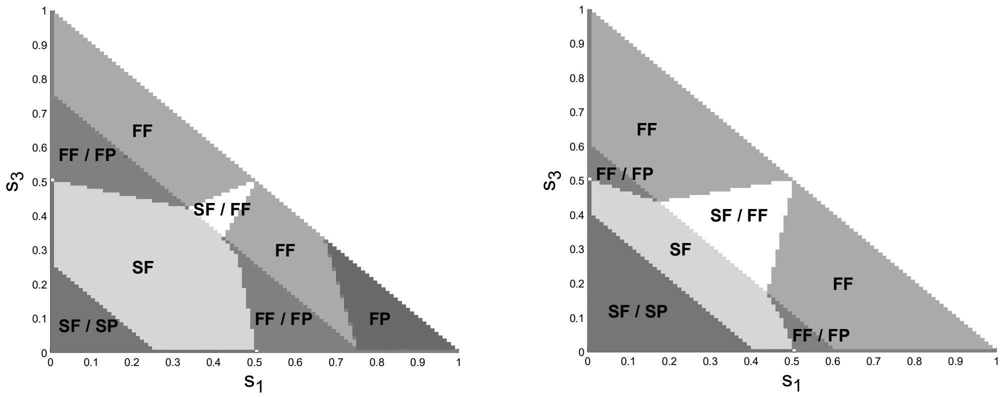
Given the work content on stations and the work velocities of workers, a policy that maximizes the long-run average throughput of the system is called the *best policy for the system*. Figure 3.6(a) shows the best policies for all work-content distributions if $v_{\min}/v_{\max} = 1/3$. Note that boundaries in Figure 3.6 can be derived according to Theorem 1. Since expressions of those boundaries are very complicated, and since some boundaries may disappear for certain $v_{\min}/v_{\max}$, we only plot the figures numerically.FP policy dominates the bottom-right domain of the graph (where s_1 is large and s_3 is small). This domain disappears if $v_{\min}/v_{\max} \geq 1/2$. Figure 3.6(b) gives anexample for $v_{\min}/v_{\max} = 2/3$. Figure 3.6 shows which policy to adopt to maximize system productivity given the work content on stations and the work velocities of workers. We observe that if $v_{\min}/v_{\max} < 1/2$ (such as Figure 3.6(a)), the SF, FF, or FP policy gives the highest throughput for any work-content distribution. If $v_{\min}/v_{\max} \geq 1/2$ (such as Figure 3.6(b)), then the SF or FF policy is the best.

Since the domains dominated by each policy in Figure 3.6 depend on the velocities of workers, one needs to reconstruct the figure if the velocities change. In practice, it is more convenient to stick to one policy for a large domain of work-content distributions, independent of workers' velocities. Note that in Figure 3.6 the SF policy is the best for most of the work-content distributions within the domain: $s_1 \leq 0.5$ and $s_3 \leq 0.5$, whereas the FF policy dominates in most of other domains. This leads to the following simple heuristic for the manager:

> *If $s_1 \leq 0.5$ and $s_3 \leq 0.5$, then use the SFpolicy; otherwise, use the FF policy.*

To implement this heuristic in practice, we first fully cross-train both workers. If $s_1 \leq 0.5$ and $s_3 \leq 0.5$, we sequence them from slowest to fastest. Otherwise, we sequence them from fastest to slowest.

Figure 3.7(a) shows the percentage of the entire feasible work-content region in which the heuristic correctly selects the best policies for various values of $v_{\min}/v_{\max}$. The heuristic is consistent with the best policies for at least 82% of the entire feasible work-content region. This consistency percentage increases to about 92% and attains 100% as $v_{\min}/v_{\max}$ approaches 0.1 and 1 respectively. The consistency percentage has a "jump" at $v_{\min}/v_{\max} = 0.5$. This is because when $v_{\min}/v_{\max} < 0.5$ the heuristic does not select the correct policy for the bottom-right corner of Figure



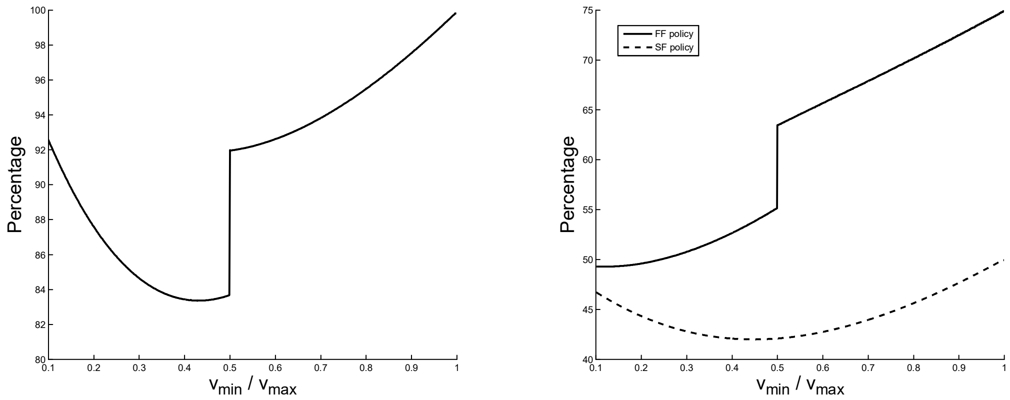
(a) $v_{\min}/v_{\max} = 1/3$ (b) $v_{\min}/v_{\max} = 2/3$

Figure 3.6: **The best policies for the system**. (a) If $v_{\min}/v_{\max} < 1/2$, the SF, FF, or FP policy gives the highest throughput for any work-content distribution. (b) If $v_{\min}/v_{\max} \geq 1/2$, the SF or FF policy gives the highest throughput.

3.6(a) where the FP policy is the best for the system.

Figure 3.7(b) compares the performance of the SF and FF policies. The dashed and solid lines represent the percentages of the entire feasible work-content region where the SF and FF policies, respectively, are the best. The percentage of the FF policy also has a gap at $v_{\min}/v_{\max} = 0.5$ due to the same reason as for Figure 3.7(a). Note that given the velocities of workers, the FF policy is the best for significantly more work-content distributions than the SF policy.

In the U-line with discrete work stations, ordering workers from slowest to fastest along the horizontal axis may not be productive if s_1 or s_3 is relatively large (that is, if work content is heavily concentrated on one "wing" of the U-line). These work-content distributions correspond to the bottom-right andtop-left corners of the feasible work-content region. This is because under the slowest-to-fastest ordering the slower worker works on station 1 or 3 with large work content, causing the faster



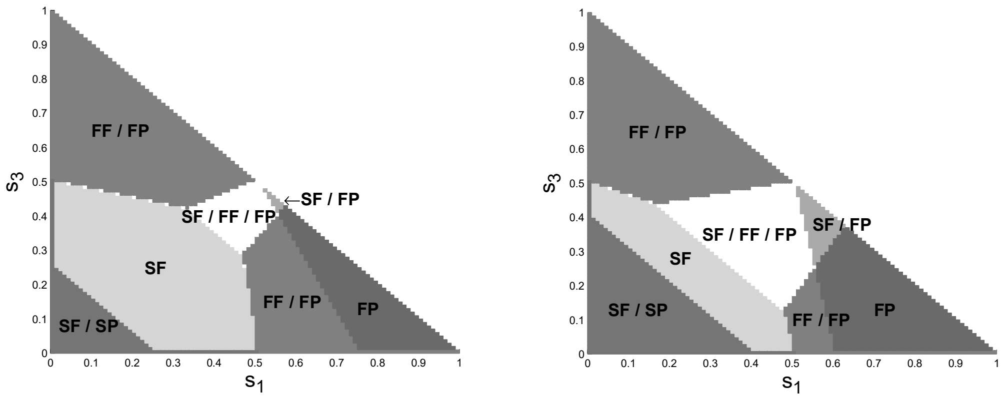
(a) (b)

Figure 3.7: **Performance of the heuristic, SF, and FF policies**. (a) The heuristic attains the maximum throughput in at least 82% of the entire feasible work-content region. (b) The FF policy dominates in a larger domain than the SF policy.

worker halted or blocked for a long time. Fortunately, this problem can be solved by using the heuristic. Comparing Figures 3.7(a) and (b), it is obvious that the heuristic performs much better than both SF and FF policies.

3.5.2 Preferredpolicies for individuals

We also analyze the preference of each worker among the four policies in Table 3.1. Figure 3.8 shows the policies that give the faster worker the highest remuneration rates for different work-content distributions. Note that the SF and FP policies dominate. This makes sense as worker 2 is faster in the slowest-to-fastest ordering. He prefers the fully cross-trained team because he earns more if he can work on stations 1 and 3. If workers are sequenced from fastest to slowest, worker 1 is faster and he prefers to restrict his colleague to work only on station 2.



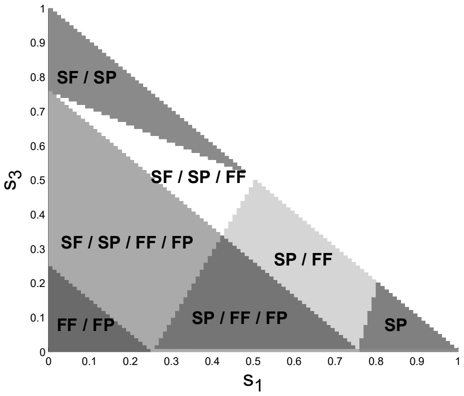
(a) $v_{\min}/v_{\max} = 1/3$ (b) $v_{\min}/v_{\max} = 2/3$

Figure 3.8: **Faster worker's preferred policies.** The faster worker prefers the SF or FP policy, which gives him the highest remuneration rate for any work-content distribution.

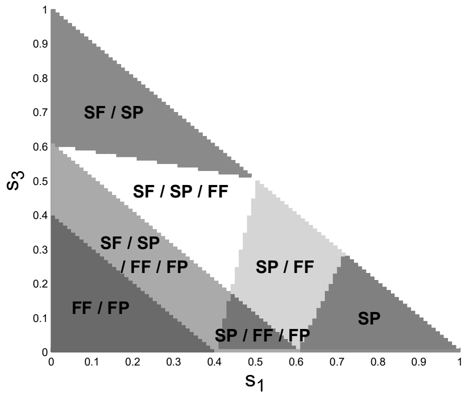
Figure 3.9 shows the policies that give the slower worker the highest remuneration rates for different work-content distributions. For any work-content distribution the FF or SP policy dominates. If workers are sequenced from fastest to slowest, worker 2 is slower and he prefers to be cross-trained to work on stations 1 and 3 so that he earns more. In the slowest-to-fastest ordering, worker 1 is slower and he prefers to restrict his colleague to work only on station 2. Figures 3.8 and 3.9 echo the results of Theorem 3: Worker 1 always prefers the partially cross-trained team, while worker 2 always prefers the fully cross-trained team.

3.5.3 Conflict andconsistency

Which policy should we choose if the system, the faster worker, andthe slower worker all have different preferences for a work-content distribution? For example, when $v_{\min} = 2$, $v_{\max} = 3$, $s_1 = 0.8$, and $s_2 = s_3 = 0.1$, Figures 3.6, 3.8, and 3.9 show



(a) $v_{\min}/v_{\max} = 1/3$



(b) $v_{\min}/v_{\max} = 2/3$

Figure 3.9: **Slower worker's preferred policies.** The slower worker prefers the FF or SP policy, which gives him the highest remuneration rate for any work-content distribution.

that the system, the faster worker, and the slower worker prefer the FF, FP, and SPpolicies respectively. Note that both workers want to be worker 1 and prefer to restrict their colleague to work only on station 2.

Under a policy π, let $\mathcal{T}^{\pi}$ be the system's throughput and let R_{s}^{π} and R_{f}^{π} denote the remuneration rates of the slower and faster workers respectively. For the above example, the system prefers the FF policy that results in $\mathcal{T}^{\mathrm{FF}} = 3.53$, $R_{\mathrm{f}}^{\mathrm{FF}} = 2.47$, and $R_{\mathrm{s}}^{\mathrm{FF}} = 1.06$. The faster worker prefers the FP policy that results in $\mathcal{T}^{\mathrm{FP}} = 3.33$, $R_{\mathrm{f}}^{\mathrm{FP}} = 3$, and $R_{\mathrm{s}}^{\mathrm{FP}} = 0.33$, but the slower worker prefers the SP policy that results in $\mathcal{T}^{\mathrm{SP}} = 2.22$, $R_{\mathrm{f}}^{\mathrm{SP}} = 0.22$, and $R_{\mathrm{s}}^{\mathrm{SP}} = 2$. We can see that under the system's preferred policy, the remuneration rate of a worker is greater than his remuneration rate under the other worker's preferred policy: $R_{\mathrm{f}}^{\mathrm{SP}} < R_{\mathrm{f}}^{\mathrm{FF}} < R_{\mathrm{f}}^{\mathrm{FP}}$ and $R_{\mathrm{s}}^{\mathrm{FP}} < R_{\mathrm{s}}^{\mathrm{FF}} < R_{\mathrm{s}}^{\mathrm{SP}}$. This observation is generalized in Theorem 4.

Theorem 4. *For any work-content distribution, suppose π^* is a policy that maxi-*

mizes the system's throughput and the faster and slower workers prefer policies π_f *and* π_s *respectively. Then* $R_s^{\pi^*} \geq R_s^{\pi_f}$ *and* $R_f^{\pi^*} \geq R_f^{\pi_s}$.

Proof. Since

$$\begin{aligned} \mathcal{T}^{\pi} &= R_{\mathrm{s}}^{\pi} + R_{\mathrm{f}}^{\pi},\ \forall \pi; \\ \mathcal{T}^{\pi^*} &\geq \mathcal{T}^{\pi_i},\ i \in \{\mathrm{s}, \mathrm{f}\}; \\ R_i^{\pi_i} &\geq R_i^{\pi^*},\ i \in \{\mathrm{s}, \mathrm{f}\}; \end{aligned}$$

we have $R_{\mathrm{s}}^{\pi^*} \geq R_{\mathrm{s}}^{\pi_{\mathrm{f}}}$ and $R_{\mathrm{f}}^{\pi^*} \geq R_{\mathrm{f}}^{\pi_{\mathrm{s}}}$. □

Theorem 4 shows that if the policy preferred by the system is adopted, then neither worker does as bad as he would under the policy preferred by his colleague. While Theorem 4 suggests a way to handle tripartite conflicts, Theorem 5 provides a way to find a policy that is preferred by all parties for those work-content distributions where both workers have preferred policies in common.

Theorem 5. *For any work-content distribution, any policy that is preferred by both faster and slower workers maximizes the system's throughput.*

Proof. We prove by contradiction. For any work-content distribution, suppose there is a policy π_0 that is preferred by both faster andslower workers but does not maximize the system's throughput. Let $\pi^* \neq \pi_0$ be a policy that results in maximum system's throughput. Due to our assumption, $R_{\mathrm{f}}^{\pi_0} \geq R_{\mathrm{f}}^{\pi^*}$ and $R_{\mathrm{s}}^{\pi_0} \geq R_{\mathrm{s}}^{\pi^*}$. We have

$$\mathcal{T}^{\pi_0} = R_{\mathrm{f}}^{\pi_0} + R_{\mathrm{s}}^{\pi_0} \geq R_{\mathrm{f}}^{\pi^*} + R_{\mathrm{s}}^{\pi^*} = \mathcal{T}^{\pi^*}.$$

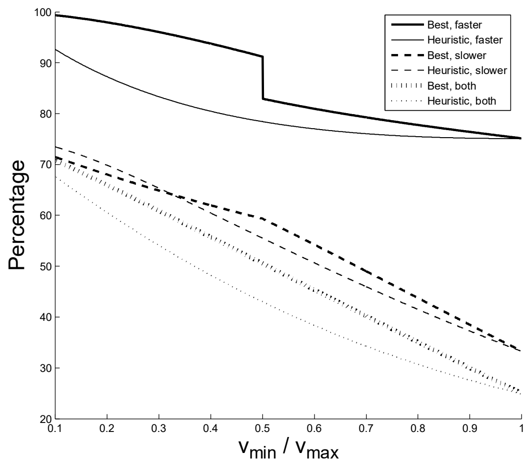


Figure 3.10: **Consistency of system's and individuals' preferences.** This figure shows the percentages of work-content distributions where each worker's preference is consistent with the best policies and the heuristic.

Since π_0 does not maximize the system's throughput, we have $\mathcal{T}^{\pi_0} < \mathcal{T}^{\pi^*}$, which leads to a contradiction. Thus, any policy that is preferred by both workers maximizes the system's throughput. □

The theorem suggests that for any work-content distributions where both workers have preferred policies in common, the manager can adopt those common policies, which will be consistent with the system's preference.

Figure 3.10 shows the consistencyof each worker's preference withthe best policies and the heuristic. The bold and the thin solid lines show the percentages of the entire feasible work-content region where the faster worker's preference is consistent with the best policies and the heuristic respectively. As shown in Figure 3.6, there could be more than one policy that gives the highest throughput for any work-content distribution. In that case we choose a policy, if any, that is consistent with the faster worker's preference.

Similarly, the bold dashed and the thin dashed lines show the percentages where the slower worker's preference is consistent with the best policies and the heuristic respectively. Finally, the bold and the thin dotted lines show the percentages where both workers' common preference is consistent with the best policies and the heuristic respectively.

Figure 3.10 shows that the faster worker's preference is consistent with the best policies for at least 75% of the entire feasible region. In contrast, the consistency between the slower worker's preference and the best policies is much lower. This implies that given two workers, the manager should pay more attention to the faster worker's preference because it maximizes system productivity for most work-content distributions. This is especially so if the workers have very different velocities (if $v_{\min}/v_{\max}$ is small). Inaddition, as workers' velocities deviate from each other, the system's preference becomes more consistent not only with individual workers' preferences, but also with both workers' common preference.

As suggested by Figures 3.7(a), the heuristic maximizes system productivity for most work-content distributions. Figure 3.10 shows that the curves corresponding to the heuristic are close to those corresponding to the best policies. These observations suggest that the heuristic performs well in terms of maximizing system productivity and that the consistency of the heuristic and workers' preferences is similar to the consistency of the best policies and workers' preferences.

To complement the studies of the heuristic and the consistencyof system's

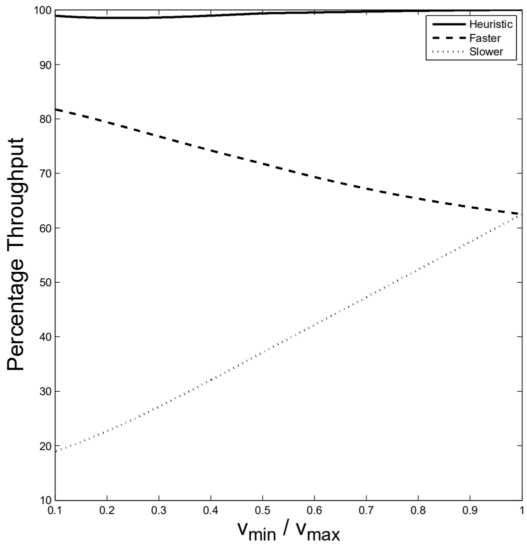


Figure 3.11: **Average percentage throughput over all work-content distributions.** The average percentage throughput of the heuristic is nearly 100%, but the average percentage throughput of the faster worker and slower worker is much lower.

and individuals' preferences, we introduce percentage throughput: The percentage throughput of the heuristic, faster worker, and slower worker is the average ratio (in percentage) of throughput under the heuristic, the faster worker's preference, and the slower worker's preference to throughput under the best policies respectively. Figure 3.11 shows how well the heuristic and workers' preferred policies perform compared to the best policies on average. Consistent with Figure 3.7(a), on average the throughput of the heuristic is almost the same as that of the best policies. Similar to Figure 3.10, the faster worker's preferred policy performs better than slower worker's, but the difference decreases in $v_{\min}/v_{\max}$. In contrary to Figure 3.10, as workers' velocities deviate from each other, the system's preferences become more consistent with faster workers' preferences, but less consistent with slower workers' preferences.

3.6 Conclusions

In this chapter we propose simple rules to coordinate workers on a three-station, two-worker U-line. If workers are fully cross-trained, the system always converges to a fixed point or a period-2 orbit. The fully cross-trained team is more productive than the partially cross-trained team for all situations, except the case where worker 1 is sufficiently faster than worker 2 ($v_1/v_2 > 2$) and the work content of station 1 is relatively large. In this exceptional case, the slower worker (worker 2) constantly blocks the faster worker (worker 1) in the fully cross-trained team, leading to low productivity.

We assume each worker is rewarded according to his long-run average contribution to the system's throughput. Under this remuneration scheme, workers prefer to be busy all the time. As a result, worker 1 always prefers the partially cross-trained team while worker 2 always prefers the fully cross-trained team.

We define a policy as a combination of the ordering of workers and the extent of cross-training. We find that the SF or FF policy maximizes the system's throughput for most work-content distributions.However, there is no single policy that dominates the entire feasible work-content region.

Fortunately, we find a heuristic that works well for practical purposes: Use the SF policy if $s_1 \leq 0.5$ and $s_3 \leq 0.5$, and use the FF policy otherwise. This heuristic is easy to implement in practice as it does not depend on workers' velocities. Despite its simplicity, the heuristic is consistent with the best policies for at least 82% of the feasible work-content region.The heuristic performs especially well when the workvelocities of workers are sufficientlysimilar ($v_{\min}/v_{\max} > 0.5$) or very different ($v_{\min}/v_{\max} < 0.14$) as the percentage of consistency with the best policies in these situations exceeds 90%. Furthermore, as the workers' work velocities become extremely close ($v_{\min}/v_{\max} \to 1$) the heuristic maximizes the system's throughput for almost all work-content distributions.

For any work-content distribution, the faster worker prefers the SF or FP policy while the slower worker prefers the FF or SP policy. The policies preferredby the system, the faster worker, and the slower worker can be all different. In that case, we finda way to resolve the tripartite conflict: Choose a policy preferred by the system as the resultant remuneration rate of each worker is at least equal to his remuneration rate under his colleague's preferred policy. Thus, choosing a system's preferred policy maximizes system productivity without making any worker extremely unhappy.

For any work-content distribution, a policy preferred by both workers maximizes the system's throughput. Figure 3.10 suggests that as workers' workvelocities deviate from each other, more such work-content distributions exist.

Between the faster and the slower workers, the system's preference is consistent with the faster worker's preference for significantly more work-content distributions. Specifically, if workers' work velocities are sufficiently different ($v_{\min}/v_{\max} \leq 0.5$), then the faster worker's preferredpolicy maximizes the system's throughput for more than 90% of the feasible work-content region.This implies that if workers have very different work velocities, the manager can adopt the policy preferred by the faster worker as it will maximize system productivity for most work-content distributions.

However, determining the system's or the faster worker's preference can be quite complicated given a work-content distribution. Furthermore, their preferences may change with the work velocities of workers. A simpler solution is to use the heuristic as it approximates the best policies well for maximizing system productivity.

Chapter 4

System productivity onan M-station U-line with a general work velocity setting

4.1 Introduction

Consider a U-shaped assembly line with M stations shown in Figure 4.1. There are three stages in the U-line. Stages 1 and 3 are separated by an aisle and stage 2 spans across the aisle. Stage 1 consists of m_1 stations $S_1(1), \ldots, S_1(m_1)$ located on one side of the aisle. Stage 2 has m_2 stations $S_2(1), \ldots, S_2(m_2)$ located across the aisle. Stage 3 consists of m_3 stations $S_3(1), \ldots, S_3(m_3)$ located on the other side of the aisle. We assume $m_1 + m_2 + m_3 = M$. Each item (an instance of the product) is initiated at the start of $S_1(1)$ and is progressively assembled in the same sequence of stations until it is completed at the end of $S_3(m_3)$.

We consider a team of two workers W_1 and W_2. W_i is cross-trained to work on *zone* Z_i — a set of stations on the line, for $i = 1, 2$. We assume only W_2 is qualified to work in stage 2 due to special skill requirements. Both W_1 and W_2 can work in stages 1 and 3. Thus, we have $Z_1 = \{S_j(k) : k = 1, \ldots, m_j; j = 1, 3\}$ and $Z_2 = \{S_j(k) : k = 1, \ldots, m_j; j = 1, 2, 3\}$. Each worker assembles only a single item at a time. At most one worker is allowed to work on a station at any time (for example, due to limited tools, equipment, or space in the station). We assume W_i works with velocity $v_{ij}(k)$ on $S_j(k) \in Z_i$. For simplicity, we neglect the time to walk from one station to another.

The U-line described above is common in manufacturing. For example, in hard drive manufacturing, eachcompleted hard drive goes through a testing process before it is shipped. This testing process typically has a U-shaped layout and is supported by two workers W_1 and W_2. The testing process consists of three main stages. In stage 1, each hard drive goes through a basic test, which is a series of simple observations that can be done by both workers. After the basic test, the hard drive enters stage 2 for an intensive test, which can only be performed by W_2 as it requires special skills. After that, the hard drive enters stage 3 where it is determined whether it is qualified for shipping. The hard drive is packaged if it passes the tests or it is rejected otherwise. In both cases, the hard drive leaves the system after the final stage. Both workers can work in stage 3.

We have seen other examples of U-lines with two workers in a molecular biology

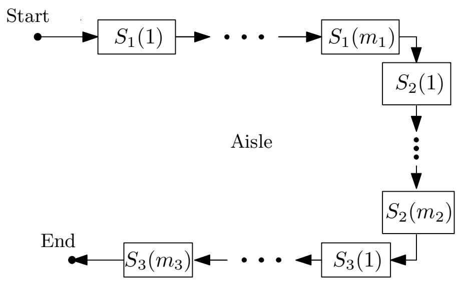


Figure 4.1: **A U-shaped assembly line.** Each item is initiated at the start of $S_1(1)$ and is progressively assembled in the same sequence of stations until it is completed at the end of $S_3(m_3)$.

lab and a microbiology lab. The former typically performs genotyping and viral load estimation, whereas the latter performs various tests related to bacteria infection and mycology. Eachitem in these examples corresponds to a batch of samples. Each sample is stored in a test tube that has a cap with a color indicating a specific analysis.

The analysis process is performed by a certified technologist (W_2) and an assistant (W_1). The layout of the process is a special case of the U-line shown in Figure 4.1 with $m_1 = m_2 = m_3 = 1$. Each batch of samples go through three stations. The first station is for sample preparation, which consists of the following sequence of tasks: (1) Centrifuge the samples. (2) Remove the caps of the test tubes so that the samples can be accessed by the probes of an auto-bioanalyzer. (3) Insert the test tubes into specialized racks of the auto-bioanalyzer.

The second station is for sample analysis, which is done by the auto-bioanalyzer. Only the technologist can work on this station as it requires a specialized license to operate the auto-bioanalyzer. This station contains tasks below that do not follow a specific sequence: (1) Perform trouble shooting in response to alarm signals of the auto-bioanalyzer (this includes renewing reagent, disposing waste liquid, and handling contingencies). (2) Tune the auto-bioanalyzer's wavelength to match the reagent's color. (3) Print and verify the report. The third station is for sample post-analysis, which consists of the following sequence of tasks: (1) Move the samples from the specialized racks to regular racks and pick up the samples with positive reaction. (2) Store the analyzed samples for rechecking.

Boththe technologist andthe assistant canwork onthe first and the third stations. At any point in time, only one worker can work on a station. This is to avoid sample contamination and is also, possibly, due to space limitation.

U-lines are adopted in the above examples because they possess several advantages over straight lines. These include providing better visibility and communications, which lead to better quality control. The travel of workers is reduced as they can execute nonconsecutive tasks that are physically close to each other especially if the aisle is narrow. Many firms adopt a U-shaped layout also because of space constraints. For a discussion on the advantages of U-lines, see Miltenburg and Wijingaard (1994).

In the U-line with two workers described above, we allow workers to dynamically share work so that the system's capacity is fully utilized subject to the constraint that only W_2 can work in stage 2. This can be challenging as it requires effective coordination of workers on the U-line. Since the professional worker W_2 is more expensive, we want him to work mainly in stage 2 that requires special skills and let the ordinary worker W_1 handles the simpler tasks in stages 1 and 3. Furthermore, the aisle is typically narrow compared to the total length of the U-line in each of the examples above. Under this setting, we want the workers to collaborate with each other without too much travel in practice. We adopt the ideas of *cellular bucket brigades* introduced by Lim (2011) to achieve the above goals.

Under the design of a cellular bucket brigade (Lim 2011), the work content of an assembly line is distributed on both sides of an aisle. Each worker works on one side when he proceeds in one direction and works on the other side when he proceeds in the reverse direction. The workers exchange work when they approach each other from opposite directions. By applying similar coordination rules, whichwill be discussed later, each item in the U-line is initiated at the start of $S_1(1)$ typically by W_1, who passes it to W_2 at some point in stage 1. W_2 then finishes the remaining work of stage 1 and continues to assemble the item in stage 2, before he passes it back to W_1 in stage 3. W_1 then completes the item at the end of $S_3(m_3)$. Our goal is to analyze the asymptotic dynamics and determine the throughput (number of items produced per unit time) of the U-line under the coordination rules proposed.

The analysis of the U-line is complicated by the facts that the number of stations in each stage can be arbitrary, different stations may have different amounts of work, workers may have different velocities on different stations, and a worker may be idle while his colleague is working on a station. We first study U-lines with three stations and two workers in Section 4.3. We define simple rules for the workers to share their work. Under these rules, we analyze the dynamics and determine the throughput of the U-line for various system settings. We then discuss the results of U-lines with

M stations and two workers in Section 4.4.

4.2 A conceptual line

Let s_j^k denote the work content of $S_j(k)$ and define $s_j = \sum_{k=1}^{m_j} s_j^k$, for $j = 1, 2, 3$. We normalize the total work content of the line such that $\sum_{j=1}^{3} s_j = 1$. The assembly line can be conceptualized as a line segment with length 1. Figure 4.2 shows such a conceptual line, which is represented by a bold solid line. The start and the end of the conceptual line are represented by points 0 and 1 respectively. The intervals $[0, s_1]$, $(s_1, s_1 + s_2]$, and $(s_1 + s_2, 1]$ correspond to the work content of stages 1, 2, and 3 respectively. The horizontal line segments $[0, s_1]$ and $(s_1 + s_2, 1]$ are parallel to each other, and the line segment $(s_1, s_1 + s_2]$ is perpendicular to them.

Define h_i as the *horizontal position* of W_i.This horizontal position is determined by projecting the *actual location* of the worker on the conceptual line to the horizontal axis. Figure 4.2 shows the relationship between the actual location and the horizontal position of each worker. To distinguish these two coordinate systems, we call any point on the conceptual line a *location* and any point on the horizontal axis a *position*.

We set the origin of the horizontal axis to be the projection of location 0 (the start of stage 1) to the axis. Note that a horizontal position can be negative if $s_1 < s_3$. Since stage 2 runs vertically across the aisle, we have $h_i \leq s_1$, for $i = 1, 2$. We require the workers to remain in a fixed sequence along the horizontal axis such

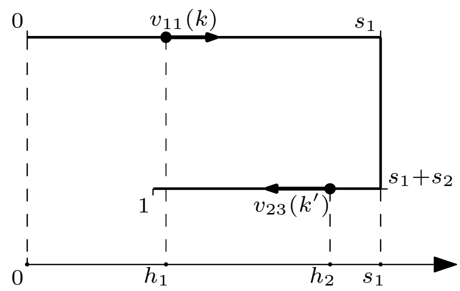


Figure 4.2: **A conceptual line.** The U-line is conceptualized as a line segment with length 1. In this graph, W_1 and W_2 work on $S_1(k)$ and $S_3(k')$ respectively. The horizontal position h_i is determined by projecting the location of worker i on the conceptual line to the horizontal axis.

that $h_1 \leq h_2$ at any point in time.

4.3 The three-station, two-worker U-lines

In this section, we discuss a special case where $m_1 = m_2 = m_3 = 1$. Thus, the notation $v_{ij}(k)$ and $S_j(k)$ can be simplified as v_{ij} and S_j, respectively, for $i = 1, 2$ and $j = 1, 2, 3$. U-lines with three stations are common in molecular biology labs andmicrobiologylabs. We fully analyze the dynamics of this special case and determine the asymptotic behavior and throughput of the system in closed-form expressions. Understanding the behavior of the three-station system will help us in the analysis of the M-station case.

4.3.1 Definitions andrules

We say W_1, who is working on S_1, *meets* W_2, who is working on S_3, when their horizontal positions coincide (that is, $h_1 = h_2$). When W_1 meets W_2, a *hand-off* between the two workers occurs: Each worker relinquishes his item, walks across the aisle, and takes over each other's item.After the hand-off, W_1 works on S_3 while W_2 proceeds on S_1.

Figure 4.3 shows how the two workers move on the U-line. Let x_n denote the n-th hand-off position. At the n-th hand-off, the two workers first relinquish their work and then walk across the aisle. After they exchange their work, W_1 works on S_3 with velocity v_{13}. When he finishes his work on S_3, he walks instantaneously to the start of S_1, initiates a new item, and works on S_1 as soon as the station is free. Meanwhile, W_2 works on S_1 with velocity v_{21}. After he reaches the end of S_1, he continues to work on S_2. W_2 then works on S_3 as soon as the station is free until he meets W_1 again at position x_{n+1}.

Specifically, the workers follow the simple rules below:

Rule for W_1:

- If you are on S_1, continue to assemble your item until you meet W_2. Then exchange work with W_2 and work on S_3.
- If you are on S_3, continue to assemble your item until you complete it. Then initiate a new item and work on S_1.

Rule for W_2:

- Continue to assemble your item along the assembly line until you meet W_1. Then exchange work with W_1 and work on S_1.

We call the above the cellular bucket brigade rules.

A worker is *blocked* if he reaches the start of a station in his zone while his colleague is still working on the station. In a three-station U-line, W_1 can be blocked only at location 0 and W_2 can be blocked only at location $s_1 + s_2$.

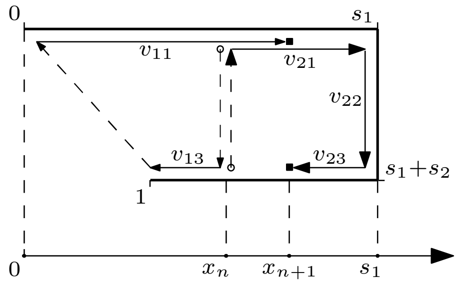


Figure 4.3: **Movement of workers on three stations.** This figure shows the paths of the two workers between the n-th and $(n+1)$-st hand-offs on a three-station U-line. The solid arrows correspond to working, while the dashed arrows correspond to instantaneous walk. The start and the end of each worker's path are represented by a circle and a square respectively.

If W_1 reaches the end of S_1 before he meets his colleague, then W_1 is *halted* at location s_1. If W_1 is halted, he remains idle until a hand-off occurs immediately after W_2 finishes his work on S_2. After the hand-off, W_1 works on S_3 while W_2 reenters S_2. On the other hand, if W_2 reaches the end of S_3 before he meets his colleague, then W_2 is halted at location 1. Note that W_2 can be halted only if $s_1 > s_3$. If W_2 is halted, he remains idle until a hand-off occurs when the horizontal positions of the two workers coincide.

4.3.2 Dynamics andthroughput

Given the stations' work content and the workers' work velocities, we determine the asymptotic dynamic behavior and the throughput of the system for any initial state. According to the cellular bucket brigade rules, if $s_1 > s_3$ then $h_1 \in [0, s_1]$ and $h_2 \in [s_1 - s_3, s_1]$.Otherwise, $h_1 \in [s_1 - s_3, s_1]$ and $h_2 \in [0, s_1]$.Thus, any hand-off position falls in the interval $I = [\max\{s_1 - s_3, 0\}, s_1]$ on the horizontal axis. Let $f : I \mapsto I$ be a function, such that $x_{n+1} = f(x_n)$. The sequence of iterates $x_1, x_2, x_3, \ldots$ is called the *orbit* of an initial iterate x_0 under f. We say x^* is a fixed point if $x^* = f(x^*)$. A *period-2 orbit* is an orbit that alternates between p and q, where $p = f(q)$ and $q = f(p)$. Note that $f(f(p)) = p$ and $f(f(q)) = q$.

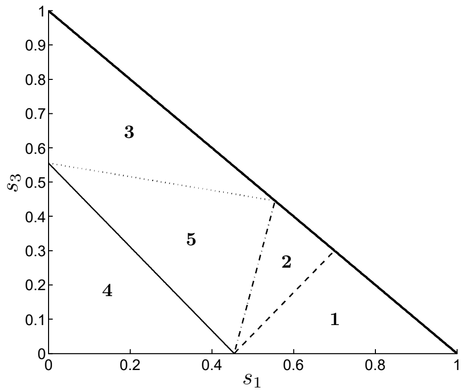
For convenience, let $\mu_{ij} = v_{22}/v_{ij}$, for $i = 1, 2$, and $j = 1, 3$. Define

$$
\begin{aligned}
\varphi &= \frac{\mu_{13} + \mu_{21}}{\mu_{11} + \mu_{23}}; \\
\eta_0 &= \frac{1 + (\mu_{13} + \mu_{21} + \mu_{23} - 1)s_1 - (\mu_{13} + 1)s_3}{\mu_{11} + \mu_{13} + \mu_{21} + \mu_{23}}; \\
\eta_1 &= \frac{1 + (\mu_{23} - 1)s_1 - s_3}{\mu_{11} + \mu_{23}}; \\
\eta_2 &= \frac{\mu_{23} s_1}{\mu_{11} + \mu_{23}}; \\
\gamma(x) &= \frac{1 + (\mu_{13} + \mu_{21} + \mu_{23} - 1)s_1 - (\mu_{13} + 1)s_3 - (\mu_{13} + \mu_{21})x}{\mu_{11} + \mu_{23}}.
\end{aligned}
$$

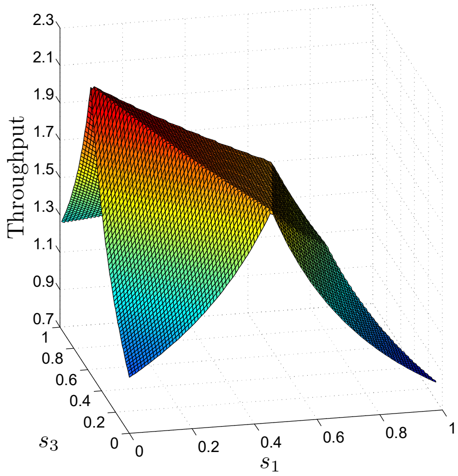
We first construct the function f and then determine the asymptotic behavior of the cellular bucket brigade by analyzing the function. We show that the system either converges to a fixed point or a period-2 orbit. We find closed-form expressions of the fixed point, the period-2 orbit, and the corresponding throughput. The details can be found in Appendix A.2. There are two cases: (1) $\varphi \leq 1$ and (2) $\varphi > 1$, which are discussed as follows.

Case 1: $\varphi \leq 1$

Figure 4.4(a) summarizes the asymptotic behavior of the two-worker cellular bucket brigade on U-lines with three workstations for $\varphi \leq 1$. Each point (s_1, s_3) in Figure 4.4(a) represents a work-content distribution on the stations. Figure 4.4(a)



(a)



(b)

Figure 4.4: **Asymptotic behaviors andthroughput ($\varphi \leq 1$).** (a) The cellular bucket brigade has different asymptotic behaviors indifferent regions. (b) The throughput in each region has a different expression. For both graphs, we set $v_{22} = 1$, $\mu_{11} = 1.2$, $\mu_{21} = 0.7$, $\mu_{13} = 0.8$, and $\mu_{23} = 1.6$.

shows that, for any given velocity ratios $\mu_{ij}, i = 1, 2,\ j = 1, 3$, the work-content distributions can be grouped into five regions. Each region corresponds to a distinct asymptotic behavior, which is summarized as follows.

Region1 (Halting andblocking): This region corresponds to systems with "long" S_1 and "short" S_3. The system converges to a fixed point $x^* = s_1 - s_3$. At the fixed point, W_2 is constantly halted at location 1 and W_1 is constantly blocked at location 0.

Region2 (Blocking): The system converges to a fixed point $x^* = \eta_1$.At the fixed point, W_1 is constantly blocked at location 0.

Region3 (Blocking): This region corresponds to systems with "short" S_1 and "long" S_3. The system converges to a fixed point $x^* = \eta_2$. At the fixed point, W_2 is constantly blocked at location $s_1 + s_2$.

Region4 (Halting): Both S_1 and S_3 are "short" in this region. The system converges to a fixed point $x^* = s_1$. At the fixed point, W_1 is constantly halted at location s_1.

Region5 (Nonidling): If $\varphi < 1$, the system converges to a fixed point $x^* = \eta_0$. If $\varphi = 1$, the system converges to a period-2 orbit: x and $\gamma(x)$, where x depends on the initial locations of the workers. Neither blocking nor halting occurs in this region.

It is noteworthy that if $\varphi < 1$, the system always converges to a fixed point in all regions. When the system operates on a fixed point, W_1 repeatedly works in a loop on the left of Figure 4.3, while W_2 covers a loop on the right that includes stage 2. Convergence to a fixed point is desirable because each worker repeats the same portion of work on each item produced. This allows workers to learn more efficiently as each of them concentrates on a set of, possibly nonconsecutive, tasks. Furthermore, each worker covers a set of tasks that are physically close to each other especially if the aisle is narrow. This reduces the travel of workers and thus may substantially boost productivity in practice. All other attractive characteristics of traditional bucket brigades on a straight-line layout are preserved under the U-line layout. For example, the system constantly seeks balance and the output is regular.

Figure 4.4(b) shows the long-run average throughput in each region. The throughput in each region has a different expression. Even though there is neither blocking nor halting, the throughput in Region 5 may be lower than that of other regions. This is because each worker has different work velocities on different stations. Thus, the assignment of stations to workers on the fixed point or the period-2 orbits may not result in the maximum throughput level.

Case 2: $\varphi > 1$

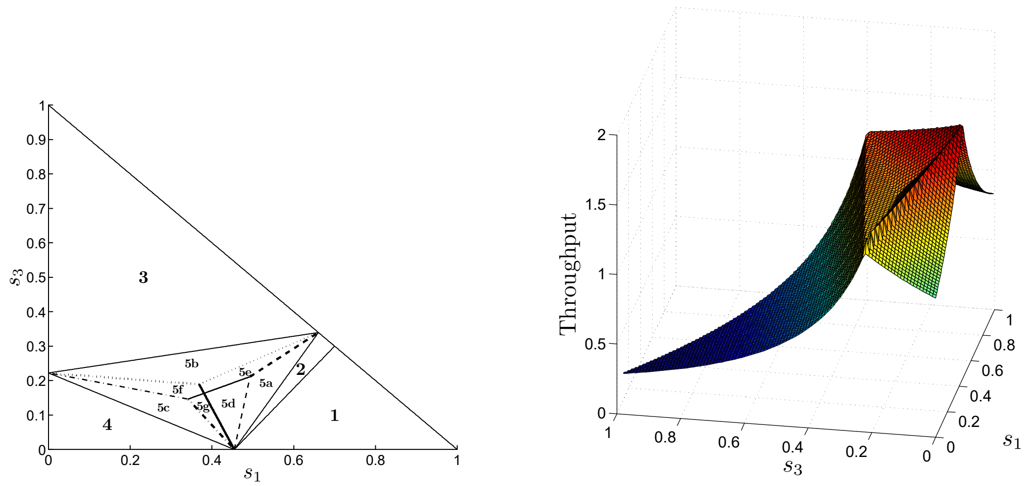
If $\varphi > 1$, the asymptotic behaviors and the expressions of throughput remain the same in all regions except for Region 5, which is now partitioned into seven subregions as shown in Figure 4.5(a). We summarize the system's asymptotic behavior in each subregion as follows.

Region5a: The system converges to a period-2 orbit: η_1 and $\gamma(\eta_1)$. At the period-2 orbit, W_1 is blocked at location 0 for every other hand-off.

Region5b: The system converges to a period-2 orbit: η_2 and $\gamma(\eta_2)$. At the period-2 orbit, W_2 is blocked at location $s_1 + s_2$ for every other hand-off.

Region5c: The system converges to a period-2 orbit: s_1 and $\gamma(s_1)$. At the period-2 orbit, W_1 is halted at location s_1 for every other hand-off.

Region5d: The system converges to a period-2 orbit: η_1 and $s_1 - s_3$. At the period-2 orbit, W_1 is blocked at location 0 for every other hand-off, and W_2 is halted at location 1 for every other hand-off.



(a) (b)

Figure 4.5: **Asymptotic behaviors and throughput ($\varphi > 1$).** (a) Region 5 is partitioned into seven subregions. (b) The throughput in each subregion of Region 5 has a different expression. For both graphs, we set $v_{22} = 1$, $\mu_{11} = 1.2$, $\mu_{21} = 0.7$, $\mu_{13} = 3.5$, and $\mu_{23} = 1.6$.

Region5e: The system converges to a period-2 orbit: η_1 and η_2. At the period-2 orbit, W_1 is blocked at location 0 for every other hand-off, and W_2 is blocked at location $s_1 + s_2$ for every other hand-off.

Region5f: The system converges to a period-2 orbit: s_1 and η_2. At the period-2 orbit, W_1 is halted at location s_1 for every other hand-off, and W_2 is blocked at location $s_1 + s_2$ for every other hand-off.

Region5g: The system converges to a period-2 orbit: s_1 and $s_1 - s_3$. At the period-2 orbit, W_1 is first blocked at location 0 and then halted at location s_1 for every other hand-off, and W_2 is halted at location 1 for every other hand-off.

Figure 4.5(b) shows that each subregion of Region 5 may have different throughput, which may be lower than that of other regions.

4.4 The M-station, two-worker U-lines

In this section we analyze the dynamics and determine the throughput of U-lines with M stations and two workers. Unlike in the three-station case, we cannot find a closed-form expression of the dynamic function f. However, we show that the system either converges to a fixedpoint or a period-2 orbit. We determine the fixed point and the corresponding throughput using an algorithmic approach. All technical details can be found in Appendix A.3.

4.4.1 Definitions andrules

Recall that a worker is in stage $j \in \{1, 2, 3\}$ if he is on station $S_j(k)$ for any $k \in \{1, \ldots, m_j\}$. We say W_1, who is working on stage 1, meets W_2, who is working on stage 3, when their horizontal positions coincide (that is, $h_1 = h_2$). There are two different types of hand-offs in the M-station U-line. A hand-off of type I occurs when W_1 meets W_2. Each worker first relinquishes his item, walks across the aisle, and then takes over each other's item. After the hand-off, W_1 works on stage 3 while W_2 proceeds on stage 1. Figure 4.6(a) shows the paths of the two workers beginning with a type I hand-off.

A type II hand-off occurs when W_1 completes an item at the end of stage 3 and the horizontal position of W_2 is negative. In this case W_1 walks back, takes over work from W_2 at the horizontal position $h_2 < 0$, and continues the work on stage 3. Meanwhile, W_2 walks across the aisle and initiates a new item in stage 1. Note that type II hand-offs are possible only if $s_1 < s_3$. Figure 4.6(b) shows the paths of the two workers beginning with a type II hand-off. Note that only type I hand-offs exist in the three-station system.

The cellular bucket brigade rules for the M-station U-line are given as follows.

Rule for W_1:

- If you are in stage 1, continue to assemble your item until you meet W_2. Then exchange work with W_2 and work in stage 3.
- If you are in stage 3, continue to assemble your item until you complete it. Upon completion,
  1. if the horizontal position of W_2 is nonnegative then initiate a new item and work in stage 1;
  2. otherwise, take over work from W_2 and continue the work in stage 3.

Rule for W_2: Continue to assemble your item along the assembly line until

- you exchange work with W_1, then work in stage 1; or
- your item is taken over by W_1, then initiate a new item and work in stage 1.

A worker is blocked if he reaches the start of a station in his zone while his colleague is still working on the station. In the M-station U-line, W_1 can be blocked at the start of any station in stage 1 and W_2 can be blocked at the start of any station in stage 3.

If W_1 reaches the end of stage 1 before he meets his colleague, then W_1 is halted at location s_1. If W_1 is halted, he remains idle until a hand-off occurs immediately

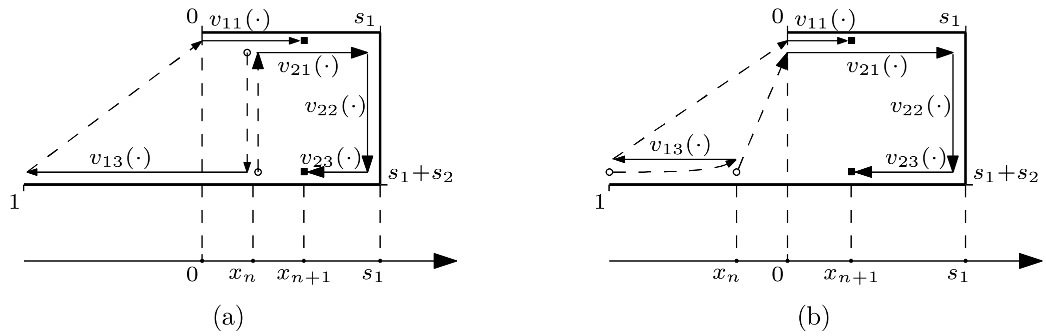


Figure 4.6: **Movement of workers on M stations.** Both graphs showthe paths of the two workers between the n-th and $(n+1)$-st hand-offs on an M-station U-line. The solid arrows correspond to working, while the dashed arrows correspond to instantaneous walk. The start and the end of each worker's path are represented by a circle and a square respectively. (a) The n-th hand-off occurs when W_1 meets W_2. (b) The n-th hand-off occurs when W_1 completes an item at location 1. He then takes over work from W_2, whose horizontal position is negative.

after W_2 finishes his work on stage 2. After the hand-off, W_1 works on stage 3 while W_2 reenters stage 2. On the other hand, if W_2 reaches the end of stage 3 before he meets his colleague, then W_2 is halted at location 1. Note that W_2 can be halted only if $s_1 > s_3$. If W_2 is halted, he remains idle until a hand-off occurs when the horizontal positions of the two workers coincide.

4.4.2 Dynamics andthroughput

According to the cellular bucket brigade rules for the M-station U-line, any hand-off position falls in the interval $I = [s_1 - s_3, s_1]$ on the horizontal axis. Note that the positions of type II hand-offs are negative. Let $f : I \mapsto I$ be a function such that $x_{n+1} = f(x_n)$, where x_n denotes the n-th hand-off position.

Due to numerous combinations of station numbers in the three stages of the U-line, we cannot enumerate each possible case and determine the function f in closed form (such as the one for the three-station case in Appendix A.2.1). However, we prove that f is continuous, non-increasing, and piecewise linear (see Appendix A.3.1). These properties of f enable us to determine the asymptotic behavior of the cellular bucket brigade on the M-station U-line.

Specifically, we show that the system has a unique fixed point x^* and has no periodic orbits of period greater than 2 (see Lemma 13 in Appendix A.3.2). We also find a sufficient condition for the fixed point x^* to be a global attractor:

Convergence Condition: For any pair of hand-off positions x and $f(x)$, one of the following cases should hold:

1. $x > 0$, $f(x) > 0$, where x falls in $S_1(k_1)$ and $S_3(k_2)$ while $f(x)$ falls in $S_1(k_3)$ and $S_3(k_4)$, and
$$\frac{1}{v_{11}(k_3)} - \frac{1}{v_{13}(k_2)} > \frac{1}{v_{21}(k_1)} - \frac{1}{v_{23}(k_4)}.$$
2. $x > 0$, $f(x) < 0$, where x falls in $S_1(k_1)$ and $S_3(k_2)$ while $f(x)$ falls in $S_3(k_4)$, and
$$-\frac{1}{v_{13}(k_2)} > \frac{1}{v_{21}(k_1)} - \frac{1}{v_{23}(k_4)}.$$
3. $x < 0$, $f(x) > 0$, where x falls in $S_3(k_2)$ while $f(x)$ falls in $S_1(k_3)$ and $S_3(k_4)$, and
$$\frac{1}{v_{11}(k_3)} - \frac{1}{v_{13}(k_2)} > -\frac{1}{v_{23}(k_4)}.$$
4. $x < 0$, $f(x) < 0$, where x falls in $S_3(k_2)$ while $f(x)$ falls in $S_3(k_4)$, and
$$-\frac{1}{v_{13}(k_2)} > -\frac{1}{v_{23}(k_4)}.$$

We show that the system always converges to the fixed point x^* if the Convergence

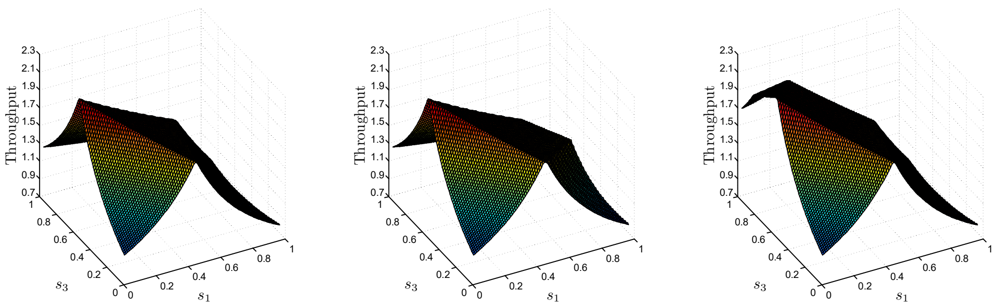
Condition holds (see Lemma 14 in Appendix A.3.2 for details). Note that if $s_1 \geq s_3$ then both x and $f(x)$ are non-negative, and so only the first inequality $1/v_{11}(k_3) - 1/v_{13}(k_2) > 1/v_{21}(k_1) - 1/v_{23}(k_4)$ should be satisfied to ensure convergence.

Checking the Convergence Condition only requires enumeration of all possible values of $k_i, i = 1, \ldots, 4$. It is straightforward to check this condition as there are at most $(m_1 + m_3)^2$ combinations of $k_i, i = 1, \ldots, 4$. Furthermore, the Convergence Condition can be checked easily if $v_{ij}(k) = v_{ij}$, where v_{ij} is a constant, for $k = 1, \ldots, m_j$.According to the rules in Section 4.3.1, hand-offpositions are always non-negative in the three-station U-line. As a result, the Convergence Condition reduces to the condition $\varphi < 1$ for the three-station case.

In addition to the Convergence Condition, we also find a necessary and sufficient conditionfor the fixed point to be a global attractor for the M-stationsystem. However, it is computationally more expensive to check this condition and we do not discuss it here. (see Lemma 15 in Appendix A.3 for details)

We develop algorithms to calculate the fixed point x^* and the throughput on the fixed point for the M-station U-line. The details of these algorithms can be found in Appendix A.3.3. We compare the throughput of systems with different work-content distributions and different numbers of stations.Since only W_2 can work in stage 2, the behavior of the system does not depend on m_2. Thus, in the following analysis we set $m_2 = 1$.

Increasing the numbers of stations in stages 1 and 3 makes the system more flexible and often results in higher throughput than the 3-station U-line. For the rest



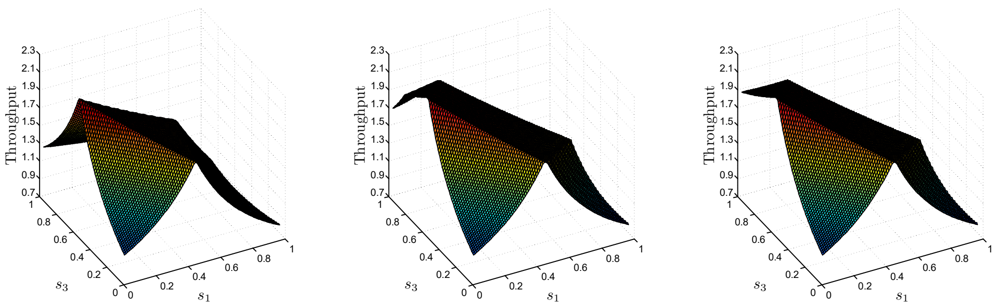
(a) $m_1 = m_2 = m_3 = 1$ (b) $m_1 = 2$ and $m_2 = m_3 = 1$ (c) $m_1 = m_2 = 1$ and $m_3 = 2$

Figure 4.7: **Throughput comparisons.** For all graphs, we set $v_{22}(1) = 1.00$, $v_{11}(k) = 0.83$, $v_{21}(k) = 1.43$, and $s_1^k = s_1/m_1$, $k = 1, \ldots, m_1$, and $v_{13}(k) = 1.25$, $v_{23}(k) = 0.63$, and $s_3^k = s_3/m_3$, $k = 1, \ldots, m_3$.

of this section, we keep the same velocity setting as that for Figure 4.4. This velocity setting satisfies the Convergence Condition. Thus, the system always converges to a fixedpoint. We first analyze the effects of increasing m_1 or m_3 individually. We thenexamine the performance of the U-line by increasing both m_1 and m_3 simultaneously.

Figure 4.7(a) shows the system's throughput with $m_1 = m_2 = m_3 = 1$ under different work-content distributions. Figure 4.7(b) shows the throughput of the same system when m_1 increases to 2. The throughput increases in Regions 1 and 2 (see Figure 4.4(a)) but remains unchanged in other regions. This is because in Regions 3, 4, and 5, even for the case where $m_1 = 1$, W_1 is not blocked at the start of stage 1 when the system operates on the corresponding fixed point. Thus, increasing m_1 does not improve the system's throughput in these regions.

Similarly, Figure 4.7(c) shows the throughput when $m_3 = 2$. Compared with Figure 4.7(a), the throughput is improved in Region 3 but remains unchanged in



(a) K = 1 (b) K = 2 (c) K = 20

Figure 4.8: **Throughput comparisons ($m_1 = m_3 = K$ and $m_2 = 1$).** For different K, the throughput remains unchanged in Regions 4 and 5, but may increase in other regions as K increases. For all graphs, we set $v_{22}(1) = 1.00$, $v_{11}(k) = 0.83$, $v_{21}(k) = 1.43$, $v_{13}(k) = 1.25$, $v_{23}(k) = 0.63$, and $s_j^k = s_j/K$, for $j = 1, 3$, $k = 1, \ldots, K$.

other regions. This is because in Regions 1, 2, 4, and 5, even for the case where $m_3 = 1$, W_2 is not blocked at the start of stage 3 when the system operates on the corresponding fixed point. Thus, the system's throughput remains unchanged in these regions as m_3 increases.

Figure 4.8 shows the system's throughput when both m_1 and m_3 simultaneously increase. Again, Figure 4.8(a) represents the base case where $m_1 = m_2 = m_3 = 1$. Figure 4.8(b) shows the throughput when m_1 and m_3 increase to 2. The throughput improvement is due to the combined effects shown in Figures 4.7(b) and (c). Figure 4.8(c) shows the throughput when m_1 and m_3 increase to 20. By comparing Figures 4.8(a) and (b), we find that there is significant improvement in throughput in Regions 1, 2, and 3 as m_1 and m_3 increase to 2. However, we see diminishing returns in Figure 4.8(c) as the numbers of stations in stages 1 and 3 continue to increase.

4.5 Conclusions

U-lines are common not only in manufacturing, but also in the healthcare industry. Inthis chapter we focus onU-lines with a professional worker and anordinary worker. Only the professional worker (W_2) is qualified to work on a critical segment of the line (stage 2), while both workers can work on the rest of the line (stages 1 and 3). We consider multiple stations in each stage. Each worker handles a single item at a time, and at most one worker is allowed to work on a station at any time. We assume worker- and station-specific work velocities.

Since the professional worker W_2 is more expensive, we want him to work mainly in stage 2 that requires special skills and let the ordinary worker W_1 handles the simpler tasks in stages 1 and 3. Furthermore, we want the workers to collaborate with each other without too much travel in practice. We propose simple rules to coordinate the two workers on the U-line such that W_2 mainly works in stage 2 and collaborates dynamically with his colleague in stages 1 and 3. Our goal is to study the asymptotic behaviors and determine the throughput of the U-line under the coordination rules proposed. The analysis of the dynamics is nontrivial due to possible blocking and halting of workers and numerous combinations of station numbers, work-content distributions, and work velocities.

We first study a three-station U-line by analyzing the asymptotic behaviors for different work-content distributions under a general velocity setting (see Figure 4.4). We then study the changes in asymptotic behavior by varying the workers' velocities (see Figure 4.5). After we fully analyze the dynamics of the three-station U-line, we extend our study to an M-station system.

For the three-stationU-line, we determine the dynamic function f inclosed form. For any given work-content distribution, the system always converges to a fixed point or a period-2 orbit. We can determine closed-form expressions of the fixed point, the period-2 orbit, and the corresponding throughput. Furthermore, we identify all cases of blocking and halting when the system operates on the fixed point or the period-2 orbit.

Convergence to a fixed point is desirable because each worker repeatedly works in the same loop on the U-line. This facilitates learning of workers. Furthermore, the travel of workers is reducedas each worker executes a set of tasks that are physically close to each other especially if the aisle is narrow. This may substantially boost productivity in practice. All other attractive characteristics of traditional bucket brigades on a straight-line layout are preserved under the U-line layout. For example, the system can restore balance after disruptions and the output is regular.

For the M-station U-line, we characterize the function f and study the asymptotic behaviors of the system. We find that, similar to the three-station case, the system converges to a fixed point or a period-2 orbit for any given number of stations in each stage and work-content distribution on the stations.We provide a sufficient condition for the fixed point to be a global attractor. This condition can be tested efficiently.

We develop algorithms to determine the fixed point and the corresponding throughput for the M-station U-line. We find that increasing the number of stations in stages 1 and 3 generally improves the throughput for certain work-content distributions (Regions 1, 2, and 3 in Figure 4.4(a)). We also observe that there are diminishing returns if we further divide the line into more stations.

Chapter 5

Conclusions

In this paper we propose simple rules to coordinate two workers on a U-line. We first analyze a three-station case with worker-specific velocities. The system always converges to a fixed point or a period-2 orbit.

Suppose that each worker is rewarded according to his long-run average contribution to the throughput. Workers always prefer to be busy, and so worker 1 prefers the partially cross-trained team but worker 2 prefers the fully cross-trained team. We define a policy as a combination of the ordering of workers and the extent of cross-training. Policy SF or FF always maximizes the system's throughput for most work-content distributions.

We find that the faster worker prefers the SF or FP policy while the slower worker prefers the FF or SP policy. The policies preferred by the system, the faster worker, and the slower worker sometimes are all different.In that situation, we can resolve the tripartite conflict by choosing a policy preferred by the system as this policy maximizes system productivity without making any worker extremely unhappy.

The system's preference is consistent with the faster worker's preference for significantly more work-content distributions. Moreover, if workers' work velocities are sufficiently different, then the faster worker's preferred policy maximizes the system's throughput for more than 90% of the feasible work-content region.

We extend our study to a three-station U-line with both worker- and station-specific velocities. We can determine the dynamic function f in closed form. For any given work-content distribution, the system always converges to a fixed point or a period-2 orbit. We can also determine closed-form expressions of the fixed point, the period-2 orbit, and the corresponding throughput.

We further generalize our research to an M-station U-line, we characterize the function f and analyze the asymptotic behaviors of the system. The system still converges to a fixed point or a period-2 orbit. We provide a sufficient condition for the fixed point to be a global attractor.

In the end, we develop efficient algorithms to compute the fixed point and the corresponding throughput for the M-station U-line. Our numerical results show that increasing the number of stations instages 1 and3 generally improves the throughput for certain work-content distributions. However, if each stage is further divided into more stations, the improvement in throughput will diminish.

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Appendix A

Technical details

A.1 Technical details for Chapter 3

A.1.1 Proof of Theorem 1

Before proving Theorem 1, we first construct the function f. Figure A.1 shows five work-content regions. Each region corresponds to a distinct form of the function f, which is determined in Lemma 1.

Lemma 1. *If $Z_1 = Z_2 = \{1, 2, 3\}$, the function f is given as follows.*

Regiona $(s_3 < -r + (r+1)s_1)$**:**

$$f(x^k) \;=\; s_1 - s_3.$$

Regionb $\left(s_3 \geq 1 - \frac{r+1}{r}s_1,\ s_3 \geq -r + (r+1)s_1,\ \textbf{and}\ s_3 < \frac{1}{r+1}s_1\right)$**:**

$$f(x^k) \;=\; \begin{cases} \frac{r}{r+1} - \frac{r}{r+1}s_3, & \textit{if } x^k \in [s_1 - s_3, s_1 - \frac{1}{r+1}s_3); \\ \frac{r}{r+1} + s_1 - s_3 - x^k, & \textit{if } x^k \in [s_1 - \frac{1}{r+1}s_3, \frac{r}{r+1}]; \\ s_1 - s_3, & \textit{otherwise.} \end{cases}$$

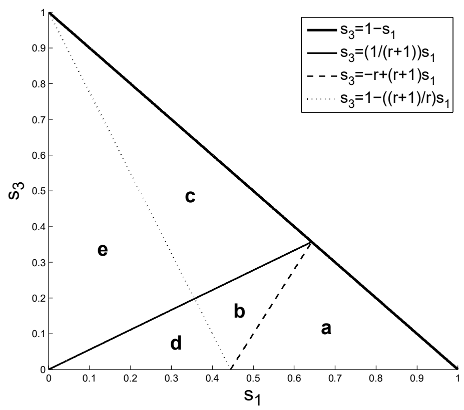

Figure A.1: **Work-content regions.** Each region corresponds to a distinct form of the function f. We set $r = 0.8$ in this example.

Regionc $\left(s_3 \geq 1 - \frac{r+1}{r}s_1 \textbf{ and } s_3 \geq \frac{1}{r+1}s_1\right)$**:**

$$f(x^k) \;=\; \begin{cases} \frac{r}{r+1} - \frac{r}{r+1}s_3, & \textit{if } x^k \in [\max(0, s_1 - s_3), s_1 - \frac{1}{r+1}s_3); \\ \frac{r}{r+1} + s_1 - s_3 - x^k, & \textit{if } x^k \in [s_1 - \frac{1}{r+1}s_3, \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3]; \\ \frac{r}{r+1}s_1, & \textit{otherwise.} \end{cases}$$

Regiond $\left(s_3 < 1 - \frac{r+1}{r}s_1 \textbf{ and } s_3 < \frac{1}{r+1}s_1\right)$**:**

$$f(x^k) \;=\; \begin{cases} s_1, & \textit{if } x^k \in [s_1 - s_3, \frac{r}{r+1} - s_3); \\ \frac{r}{r+1} + s_1 - s_3 - x^k, & \textit{if } x^k \in [\frac{r}{r+1} - s_3, \frac{r}{r+1}]; \\ s_1 - s_3, & \textit{otherwise.} \end{cases}$$

Regione $\left(s_3 < 1 - \frac{r+1}{r}s_1 \textbf{ and } s_3 \geq \frac{1}{r+1}s_1\right)$**:**

$$f(x^k) \;=\; \begin{cases} s_1, & \textit{if } x^k \in [\max(0, s_1 - s_3), \frac{r}{r+1} - s_3); \\ \frac{r}{r+1} + s_1 - s_3 - x^k, & \textit{if } x^k \in [\frac{r}{r+1} - s_3, \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3]; \\ \frac{r}{r+1}s_1, & \textit{otherwise.} \end{cases}$$

Proof. We construct the function f for the following two cases separately: (I) $s_1 > s_3$ and (II) $s_1 \leq s_3$. For Case (I), the hand-off position x^k falls in the interval $[s_1 - s_3, s_1]$ on the horizontal axis. Figure A.2 shows the conceptual line for Case (I). Note that the actual locations of workers on the line immediately *after* the k-th hand-off are shown in the figure. In this case, worker 1 may be blocked at locations 0 and s_1 on the line, and worker 2 may be blockedat location $s_1 + s_2$ and haltedat location1. We determine the next hand-off position x^{k+1} by considering all possible combinations of blocking and halting events.

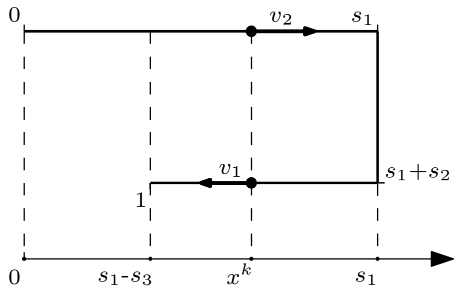

Figure A.2: **Case I ($s_1 > s_3$).** The hand-off position x^k falls in the interval $[s_1 - s_3, s_1]$ The actual locations of workers on the line immediately after the k-th hand-off are shown.

(I) $s_1 > s_3$ ($x^k \in [s_1 - s_3, s_1]$):

(A) If $\frac{s_1 - x^k}{v_2} \leq \frac{x^k - s_1 + s_3}{v_1} \Leftrightarrow x^k \geq s_1 - \frac{1}{r+1} s_3$, then worker 1 is not blocked at location 0.

(1) If $\frac{1 - s_3 - x^k}{v_2} < \frac{x^k - s_1 + s_3}{v_1} \Leftrightarrow x^k > \frac{r}{r+1} + \frac{1}{r+1} s_1 - s_3$, then worker 2 is blocked at location $s_1 + s_2$.

(a) If $\frac{s_3}{v_2} < \frac{s_1 - s_3}{v_1} \Leftrightarrow s_3 < \frac{1}{r+1} s_1$, then worker 2 is halted at location 1 and $x^{k+1} = s_1 - s_3$.

(b) If $s_3 \geq \frac{1}{r+1} s_1$, then worker 2 is not halted at location 1 and $x^{k+1} = \frac{r}{r+1} s_1$.

(2) If $x^k \leq \frac{r}{r+1} + \frac{1}{r+1} s_1 - s_3$, then worker 2 is not blocked at location $s_1 + s_2$.

(a) If $\frac{1 - x^k}{v_2} < \frac{x^k}{v_1} \Leftrightarrow x^k > \frac{r}{r+1}$, then worker 2 is halted at location 1 and $x^{k+1} = s_1 - s_3$.

(b) If $x^k \leq \frac{r}{r+1}$, then worker 2 is not halted at location 1.

(i) If $\frac{1 - s_3 - x^k}{v_2} > \frac{x^k + s_3}{v_1} \Leftrightarrow x^k < \frac{r}{r+1} - s_3$, then worker 1 is blocked at location s_1 and $x^{k+1} = s_1$.

(ii) If $x^k \geq \frac{r}{r+1} - s_3$, then worker 1 is not blocked at location s_1 and $x^{k+1} = \frac{r}{r+1} + s_1 - s_3 - x^k$.

(B) If $x^k < s_1 - \frac{1}{r+1}s_3$, then worker 1 is blocked at location 0.

(1) If $\frac{1-s_1}{v_2} < \frac{s_1-s_3}{v_1} \Leftrightarrow s_3 < -r + (r+1)s_1$, then worker 2 is halted at location 1 and $x^{k+1} = s_1 - s_3$.

(2) If $s_3 \geq -r + (r+1)s_1$, then worker 2 is not halted at location 1.

(a) If $\frac{s_1}{v_1} < \frac{1-s_1-s_3}{v_2} \Leftrightarrow s_3 < 1 - \frac{r+1}{r}s_1$, then worker 1 is blocked at location s_1 and $x^{k+1} = s_1$.

(b) If $s_3 \geq 1 - \frac{r+1}{r}s_1$, then worker 1 is not blocked at location s_1 and $x^{k+1} = \frac{r}{r+1} - \frac{r}{r+1}s_3$.

For Case (II), the hand-off position x^k falls in the interval $[0, s_1]$ on the horizontal axis. Figure A.3 shows the conceptual line for Case (II). The actual locations of workers on the line immediately after the k-th hand-off are shown in the figure. In this case, worker 1 may be blockedat locations 0 and s_1 onthe line, and worker 2 may be blocked at location $s_1 + s_2$. We determine the next hand-off position x^{k+1} by considering all possible combinations of blocking events.

(II) $s_1 \leq s_3$ ($x^k \in [0, s_1]$):

(A) If $\frac{s_1-x^k}{v_2} \leq \frac{x^k-s_1+s_3}{v_1} \Leftrightarrow x^k \geq s_1 - \frac{1}{r+1}s_3$, then worker 1 is not blocked at location 0.

(1) If $\frac{1-s_3-x^k}{v_2} < \frac{x^k-s_1+s_3}{v_1} \Leftrightarrow x^k > \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3$, then worker 2 is blocked at location $s_1 + s_2$ and $x^{k+1} = \frac{r}{r+1}s_1$.

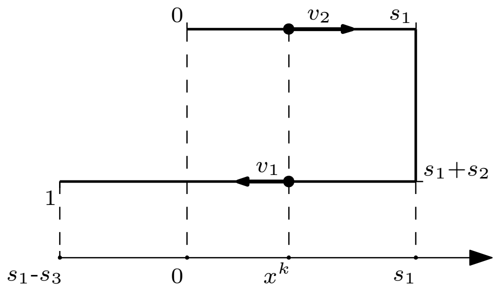

Figure A.3: **Case II ($s_1 \leq s_3$).** The hand-off position x^k falls in the interval $[0, s_1]$ on the horizontal axis. The actual locations of workers on the line immediately after the k-th hand-off are shown.

(2) If $x^k \leq \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3$, then worker 2 is not blocked at location $s_1 + s_2$.

(a) If $\frac{1-s_3-x^k}{v_2} > \frac{x^k+s_3}{v_1} \Leftrightarrow x^k < \frac{r}{r+1} - s_3$, then worker 1 is blocked at location s_1 and $x^{k+1} = s_1$.

(b) If $x^k \geq \frac{r}{r+1} - s_3$, then worker 1 is not blocked at location s_1 and $x^{k+1} = \frac{r}{r+1} + s_1 - s_3 - x^k$.

(B) If $x^k < s_1 - \frac{1}{r+1}s_3$, then worker 1 is blocked at location 0.

(1) If $\frac{s_1}{v_1} < \frac{1-s_1-s_3}{v_2} \Leftrightarrow s_3 < 1 - \frac{r+1}{r}s_1$, then worker 1 is blocked at location s_1 and $x^{k+1} = s_1$.

(2) If $s_3 \geq 1 - \frac{r+1}{r}s_1$, then worker 1 is not blocked at location s_1 and $x^{k+1} = \frac{r}{r+1} - \frac{r}{r+1}s_3$.

Now, we check the function f in each region of Figure A.1 using the above results.

Region a: Since $s_3 < -r + (r+1)s_1$, we have $s_1 - \frac{1}{r+1}s_3 > \frac{r}{r+1}$ and $s_3 < \frac{1}{r+1}s_1$ (because $s_1 + s_3 \leq 1$), and thus $s_1 > s_3$. The inequality $s_1 > s_3$ implies that this region corresponds to Case (I). If $x^k < s_1 - \frac{1}{r+1}s_3$, then we have Case (I)(B)(1): $x^{k+1} = s_1 - s_3$. Otherwise, we either have Case (I)(A)(1)(a) due to the inequality $s_3 < \frac{1}{r+1}s_1$ or Case

(I)(A)(2)(a) due to the inequalities $x^k \geq s_1 - \frac{1}{r+1}s_3 > \frac{r}{r+1}$, and thus $x^{k+1} = s_1 - s_3$. Therefore, for any x^k, we have $x^{k+1} = s_1 - s_3$.

Region b: In this region, we have $s_3 \geq 1 - \frac{r+1}{r}s_1$, $s_3 \geq -r + (r+1)s_1$, and $s_3 < \frac{1}{r+1}s_1$. The last inequality implies $s_1 > s_3$. All the three inequalities imply $\frac{r}{r+1} - s_3 \leq s_1 - \frac{1}{r+1}s_3 \leq \frac{r}{r+1} < \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3$. Since $s_1 > s_3$, this region corresponds to Case (I). If $x^k < s_1 - \frac{1}{r+1}s_3$, then we have Case (I)(B)(2)(b): $x^{k+1} = \frac{r}{r+1} - \frac{r}{r+1}s_3$. If $s_1 - \frac{1}{r+1}s_3 \leq x^k \leq \frac{r}{r+1}$, then we have Case (I)(A)(2)(b)(ii) due to the inequalities $x^k \geq s_1 - \frac{1}{r+1}s_3 \geq \frac{r}{r+1} - s_3$, and thus $x^{k+1} = \frac{r}{r+1} + s_1 - s_3 - x^k$. If $x^k > \frac{r}{r+1}$, then we either have Case (I)(A)(1)(a) due to the inequality $s_3 < \frac{1}{r+1}s_1$ or Case (I)(A)(2)(a) due to the inequality $x^k > \frac{r}{r+1}$, and thus $x^{k+1} = s_1 - s_3$.

Region c: Since $s_3 \geq 1 - \frac{r+1}{r}s_1$ and $s_3 \geq \frac{1}{r+1}s_1$, we have $\frac{r}{r+1} - s_3 \leq s_1 - \frac{1}{r+1}s_3 \leq \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3 \leq \frac{r}{r+1}$. The inequalities $s_3 \geq \frac{1}{r+1}s_1$ and $s_1 + s_3 \leq 1$ imply $s_3 \geq -r + (r+1)s_1$. Both Cases (I) and (II) are possible in this region.

For Case (I), if $x^k < s_1 - \frac{1}{r+1}s_3$, then we have Case (I)(B)(2)(b): $x^{k+1} = \frac{r}{r+1} - \frac{r}{r+1}s_3$. If $s_1 - \frac{1}{r+1}s_3 \leq x^k \leq \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3$, then we have Case (I)(A)(2)(b)(ii) due to the inequalities $\frac{r}{r+1} - s_3 \leq s_1 - \frac{1}{r+1}s_3$ and $\frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3 \leq \frac{r}{r+1}$, and thus $x^{k+1} = \frac{r}{r+1} + s_1 - s_3 - x^k$. If $x^k > \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3$, then we have Case (I)(A)(1)(b) due to the inequality $s_3 \geq \frac{1}{r+1}s_1$, and thus $x^{k+1} = \frac{r}{r+1}s_1$.

For Case (II), if $x^k < s_1 - \frac{1}{r+1}s_3$, then we have Case (II)(B)(2): $x^{k+1} = \frac{r}{r+1} - \frac{r}{r+1}s_3$. If $s_1 - \frac{1}{r+1}s_3 \leq x^k \leq \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3$, then we have Case (II)(A)(2)(b) due to the inequalities $x^k \geq s_1 - \frac{1}{r+1}s_3 \geq \frac{r}{r+1} - s_3$, and thus $x^{k+1} = \frac{r}{r+1} + s_1 - s_3 - x^k$. If $x^k > \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3$, then we have Case (II)(A)(1): $x^{k+1} = \frac{r}{r+1}s_1$.

Region d: In this region, we have $s_3 < 1 - \frac{r+1}{r}s_1$ and $s_3 < \frac{1}{r+1}s_1$. The first inequality implies $s_3 \geq -r + (r+1)s_1$ and the second inequality implies $s_1 > s_3$. Both inequalities imply $s_1 - \frac{1}{r+1}s_3 < \frac{r}{r+1} - s_3 \leq \frac{r}{r+1} < \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3$. Since $s_1 > s_3$, this region corresponds to Case (I). If $x^k < \frac{r}{r+1} - s_3$, then we either have Case (I)(B)(2)(a) due to the inequality $s_3 \geq -r + (r+1)s_1$ or Case (I)(A)(2)(b)(i) due to the inequalities $x^k < \frac{r}{r+1} - s_3 \leq \frac{r}{r+1} < \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3$, and thus $x^{k+1} = s_1$. If $\frac{r}{r+1} - s_3 \leq x^k \leq \frac{r}{r+1}$, then we have Case (I)(A)(2)(b)(ii) due to the inequalities $x^k \leq \frac{r}{r+1} < \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3$, and thus $x^{k+1} = \frac{r}{r+1} + s_1 - s_3 - x^k$. If $x^k > \frac{r}{r+1}$, then we either have Case (I)(A)(1)(a) due to the inequality $s_3 < \frac{1}{r+1}s_1$ or Case (I)(A)(2)(a), and thus $x^{k+1} = s_1 - s_3$.

Region e: Since $s_3 < 1 - \frac{r+1}{r}s_1$ and $s_3 \geq \frac{1}{r+1}s_1$, we have $s_3 \geq -r + (r+1)s_1$ and $s_1 - \frac{1}{r+1}s_3 < \frac{r}{r+1} - s_3 \leq \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3 \leq \frac{r}{r+1}$. Both Cases (I) and (II) are possible in this region.

For Case (I), if $x^k < \frac{r}{r+1} - s_3$, then we either haveCase (I)(B)(2)(a) or Case (I)(A)(2)(b)(i) due to the inequalities $x^k < \frac{r}{r+1} - s_3 \leq \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3 \leq \frac{r}{r+1}$, and thus $x^{k+1} = s_1$. If $\frac{r}{r+1} - s_3 \leq x^k \leq \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3$, then we have Case (I)(A)(2)(b)(ii): $x^{k+1} = \frac{r}{r+1} + s_1 - s_3 - x^k$. If $x^k > \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3$, then we have Case (I)(A)(1)(b) due to the inequality $s_3 \geq \frac{1}{r+1}s_1$, and thus $x^{k+1} = \frac{r}{r+1}s_1$.

For Case (II), if $x^k < \frac{r}{r+1} - s_3$, then we either have Case (II)(B)(1) due tothe inequality $s_3 < 1 - \frac{r+1}{r}s_1$ or Case (II)(A)(2)(a) due to the inequalities $x^k < \frac{r}{r+1} - s_3 \leq \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3$, and thus $x^{k+1} = s_1$. If $\frac{r}{r+1} - s_3 \leq x^k \leq \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3$, then we have Case (II)(A)(2)(b): $x^{k+1} = \frac{r}{r+1} + s_1 - s_3 - x^k$. If $x^k > \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3$, then we have Case (II)(A)(1): $x^{k+1} = \frac{r}{r+1}s_1$. □

We also need the following lemma to prove Theorem 1.

Lemma 2. *Suppose* $x^{k+1} = g(x^k)$*, where* $g : [A, B] \mapsto [A, B]$ $(0 \leq A < B)$ *has the* *following form*

$$g(x) = \begin{cases} Y, & \textit{if } x \in [A, X_1); \\ Y + X_1 - x, & \textit{if } x \in [X_1, X_2]; \\ Y + X_1 - X_2, & \textit{otherwise.} \end{cases}$$

If $Y \leq X_1$, then the system converges to a fixed point Y. If $X_1 < Y < 2X_2 - X_1$, then the system converges to a period-2 orbit. Otherwise, the system converges to a fixed point $Y + X_1 - X_2$.

Proof. If $Y \leq X_1$, then Y is a fixed point. For any initial point $x \in [A, B]$, we have $g(x) \leq Y \leq X_1$. So $g(g(x)) = Y$, and the system stays at the fixed point Y. Similarly, if $Y \geq 2X_2 - X_1$, then $Y + X_1 - X_2$ is a fixed point. For any initial point $x \in [A, B]$, we have $g(g(x)) = Y + X_1 - X_2$, and the system stays at the fixed point $Y + X_1 - X_2$.

If $X_1 < Y < 2X_2 - X_1$, then for any initial point $x \in [A, B]$, there are three possible cases:

1. If $x < X_1$, then $g(x) = Y$.

   - If $X_1 < Y \leq X_2$, then $g(Y) = Y + X_1 - Y = X_1$. Thus, $g(g(Y)) = Y$, which means the system converges to a period-2 orbit.

   - If $X_2 < Y < 2X_2 - X_1$, then $g(Y) = Y + X_1 - X_2$. We have $g(Y) \in [X_1, X_2]$. Thus, $g(g(Y)) = Y + X_1 - g(Y) = X_2$, which implies $g(g(g(Y))) = Y + X_1 - X_2 = g(Y)$. The system converges to a period-2 orbit.

2. If $x > X_2$, then $g(x) = Y + X_1 - X_2$. Let $Y' = Y + X_1 - X_2$. We have $2X_1 - X_2 < Y' < X_2$.

- If $X_1 \leq Y' < X_2$, then $g(Y') = Y + X_1 - Y' = X_2$. Thus, $g(g(Y')) = Y + X_1 - X_2 = Y'$, which means the system converges to a period-2 orbit.

- If $2X_1 - X_2 < Y' < X_1$, then $g(Y') = Y$. We have $g(Y') \in [X_1, X_2]$. Thus, $g(g(Y')) = Y + X_1 - g(Y') = X_1$, which implies $g(g(g(Y'))) = Y + X_1 - X_1 = Y = g(Y')$. The system converges to a period-2 orbit.

3. If $X_1 \leq x \leq X_2$, then $g(x) = Y + X_1 - x$.

   - If $X_1 \leq g(x) \leq X_2$, then $g(g(x)) = x$. The system converges to a period-2 orbit.

   - If $g(x) < X_1$, then $g(g(x)) = Y$ and this reduces to Case 1.

   - If $g(x) > X_2$, then $g(g(x)) = Y + X_1 - X_2$ and this reduces to Case 2.

□

Note that the function g in Lemma 2 represents a general form of the function f in Lemma 1. We use the properties of function g described in Lemma 2 to prove Theorem 1 as follows.

Proof. We partition the entire feasible work-content area into five regions shown in Figure 3.3(a). We determine the asymptotic behavior and throughput of the system in each region separately.

Region 1: This region is identical to Region a in Figure A.1. Lemma 1 shows that the system always converges to the fixed point $x^* = s_1 - s_3$. When the system operates on the fixed point, according to the proof of Lemma 1 (see Region a), worker 1 is always blocked at location 0 and worker 2 is always halted at location 1. The long-run average throughput is $\mathcal{T}^{\mathrm{F}} = \left(\frac{s_1 - x^*}{v_2} + \frac{x^*}{v_1}\right)^{-1} = \frac{v_1}{s_1 + (r-1)s_3}$.

Region 2: This region falls in Regions b and c in Figure A.1. Since $s_1 > \frac{r}{r+1} - \frac{r-1}{r+1}s_3 \Leftrightarrow \frac{r}{r+1} - \frac{r}{r+1}s_3 < s_1 - \frac{1}{r+1}s_3$, according to the function f in Regions b and c as well as Lemma 2, the system converges to the fixed point $x^* = \frac{r}{r+1} - \frac{r}{r+1}s_3$. When the system operates on the fixed point, according to the proof of Lemma 1 (see Regions b andc), worker 1 is always blockedat location 0. The long-run average throughput is $\mathcal{T}^{\mathrm{F}} = \left(\frac{s_1 - x^*}{v_2} + \frac{x^*}{v_1}\right)^{-1} = \frac{(r+1)v_2}{(r+1)s_1 + (1-r)(1-s_3)}$.

Region 3: This regionfalls in Regions c ande in Figure A.1. Since $s_3 > \frac{r}{r+1} - \frac{r-1}{r+1}s_1 \Leftrightarrow \frac{r}{r+1}s_1 > \frac{r}{r+1} + \frac{1}{r+1}s_1 - s_3$, according to the function f in Regions c and e as well as Lemma 2, the system converges to the fixed point $x^* = \frac{r}{r+1}s_1$. When the system operates on the fixed point, according to the proof of Lemma 1 (see Regions c and e), worker 2 is always blocked at location $s_1 + s_2$. The long-run average throughput is $\mathcal{T}^{\mathrm{F}} = \left(\frac{s_3 - s_1 + x^*}{v_1} + \frac{x^*}{v_1}\right)^{-1} = \frac{(r+1)v_1}{(r+1)s_3 + (r-1)s_1}$.

Region 4: Since $s_3 < \frac{r}{r+1} - s_1 \Leftrightarrow \frac{s_1 + s_3}{v_1} < \frac{1 - s_1 - s_3}{v_2}$, worker 1 is always blocked at location s_1. The system converges to the fixedpoint $x^* = s_1$. The long-runaverage throughput is $\mathcal{T}^{\mathrm{F}} = \left(\frac{1 - s_1 - s_3}{v_2}\right)^{-1} = \frac{v_2}{1 - s_1 - s_3}$.

Region 5: This region falls in Regions b, c, d, and e in Figure A.1. According to the function f in Regions b, c, d, and e as well as Lemma 2, it can be shown that the system converges to a period-2 orbit that comprises of points x and $\frac{r}{r+1} + s_1 - s_3 - x$, where x depends on the initial locations of the workers. When the system operates on the period-2 orbit, according to the proof of Lemma 1 (see Regions b, c, d, and e), neither blocking nor halting occurs. The system fully uses its production capacity and thus the long-run average throughput is $\mathcal{T}^{\mathrm{F}} = v_1 + v_2$. □

A.1.2 Proof of Corollary1

We provide a detailed comparison on the throughput of the fully cross-trained team with that of the partially cross-trained team in the following lemma, from which Corollary 1 immediately follows.

Lemma 3. *The long-run average throughput of the fully cross-trained team ($\mathcal{T}^F$) is compared with that of the partially cross-trained team ($\mathcal{T}^P$) in each region of Figure 3.3(a) as follows.*

Region1: *If $r < 2$, then $\mathcal{T}^F > \mathcal{T}^P$. If $r = 2$, then $\mathcal{T}^F = \mathcal{T}^P$. Otherwise, $\mathcal{T}^F < \mathcal{T}^P$.*

Region2: *If $r < 2$, then $\mathcal{T}^F > \mathcal{T}^P$. If $r = 2$, then $\mathcal{T}^F > \mathcal{T}^P$ except at the boundary $s_3 = -r + (r+1)s_1$ where $\mathcal{T}^F = \mathcal{T}^P$. Otherwise, we have the following three cases:*

1. *If $s_1 < \frac{r}{r+1} + \left(\frac{1}{r-1} - \frac{r}{r+1}\right) s_3$, then $\mathcal{T}^F > \mathcal{T}^P$.*
2. *If $s_1 = \frac{r}{r+1} + \left(\frac{1}{r-1} - \frac{r}{r+1}\right) s_3$, then $\mathcal{T}^F = \mathcal{T}^P$.*
3. *If $s_1 > \frac{r}{r+1} + \left(\frac{1}{r-1} - \frac{r}{r+1}\right) s_3$, then $\mathcal{T}^F < \mathcal{T}^P$.*

Region3: *$\mathcal{T}^F > \mathcal{T}^P$.*

Region4: *$\mathcal{T}^F = \mathcal{T}^P$.*

Region5: *$\mathcal{T}^F > \mathcal{T}^P$ except at the boundary $s_3 = \frac{r}{r+1} - s_1$ where $\mathcal{T}^F = \mathcal{T}^P$.*

Proof. According to Theorems 1 and 2, $\mathcal{T}^{\mathrm{F}} = \mathcal{T}^{\mathrm{P}}$ in Region 4 of Figure 3.3(a). In other regions of Figure 3.3(a), we have $\mathcal{T}^{\mathrm{P}} = \frac{v_1}{s_1+s_3}$. We will compare $\mathcal{T}^{\mathrm{P}}$ with $\mathcal{T}^{\mathrm{F}}$ (determined in Theorem 1) in each of these regions.

Region 1: In this region $\mathcal{T}^{\mathrm{F}} = \frac{v_1}{s_1+(r-1)s_3}$ is greater than, equal to, or less than $\mathcal{T}^{\mathrm{P}}$ if r is less than, equal to, or greater than 2 respectively.

Region 2: In this region $\mathcal{T}^{\mathrm{F}} = \frac{(r+1)v_2}{(r+1)s_1+(1-r)(1-s_3)}$. There are three cases:

1. If $r < 1$, then $\mathcal{T}^{\mathrm{P}} < \mathcal{T}^{\mathrm{F}}$ if $s_1 > \frac{r}{r+1} + \left(\frac{1}{r-1} - \frac{r}{r+1}\right) s_3$. In Region 2, $s_1 > \frac{r}{r+1} - \frac{r-1}{r+1} s_3 > \frac{r}{r+1} + \left(\frac{1}{r-1} - \frac{r}{r+1}\right) s_3$. Thus, we have $\mathcal{T}^{\mathrm{P}} < \mathcal{T}^{\mathrm{F}}$.

2. If $r = 1$, then $\mathcal{T}^{\mathrm{P}} = \frac{v_1}{s_1+s_3} < \frac{v_1}{s_1} = \frac{(r+1)v_2}{(r+1)s_1+(1-r)(1-s_3)} = \mathcal{T}^{\mathrm{F}}$.

3. If $r > 1$, $\mathcal{T}^{\mathrm{P}}$ is less than, equal to, or greater than $\mathcal{T}^{\mathrm{F}}$ if s_1 is less than, equal to, or greater than $\frac{r}{r+1} + \left(\frac{1}{r-1} - \frac{r}{r+1}\right) s_3$ respectively. In Region 2, we have $s_3 \geq -r + (r+1)s_1 \Rightarrow s_1 \leq \frac{r}{r+1} + \frac{1}{r+1} s_3$. If $1 < r \leq 2$, then $s_1 \leq \frac{r}{r+1} + \frac{1}{r+1} s_3 \leq \frac{r}{r+1} + \left(\frac{1}{r-1} - \frac{r}{r+1}\right) s_3$, which implies $\mathcal{T}^{\mathrm{P}} \leq \mathcal{T}^{\mathrm{F}}$. Note that $\mathcal{T}^{\mathrm{P}} = \mathcal{T}^{\mathrm{F}}$ if and only if $r = 2$ and $s_3 = -r + (r+1)s_1$.

Therefore in this region, if $r \leq 2$, then $\mathcal{T}^{\mathrm{P}} \leq \mathcal{T}^{\mathrm{F}}$, and the equality holds if and only if $r = 2$ and $s_3 = -r + (r+1)s_1$. Otherwise, $\mathcal{T}^{\mathrm{P}}$ is less than, equal to, or greater than $\mathcal{T}^{\mathrm{F}}$ if s_1 is less than, equal to, or greater than $\frac{r}{r+1} + \left(\frac{1}{r-1} - \frac{r}{r+1}\right) s_3$ respectively.

Region 3: In this region $\mathcal{T}^{\mathrm{P}} = \frac{v_1}{s_1+s_3} = \frac{(r+1)v_1}{(r+1)s_3+(r+1)s_1} < \frac{(r+1)v_1}{(r+1)s_3+(r-1)s_1} = \mathcal{T}^{\mathrm{F}}$.

Region 5: In this region $s_1 + s_3 \geq \frac{r}{r+1}$, so we have $\mathcal{T}^{\mathrm{P}} = \frac{v_1}{s_1+s_3} \leq v_1 + v_2 = \mathcal{T}^{\mathrm{F}}$. The equality only holds at the boundary $s_1 + s_3 = \frac{r}{r+1}$. □

A.1.3 Proof of Theorem 3

Proof. In Region 4 of Figure 3.3(a) the system has identical asymptotic behavior in both fully and partially cross-trained teams (see Theorems 1 and 2). As a result, worker i has the same remuneration rate in both teams in Region 4, for $i = 1, 2$. Therefore, we only need to analyze the preference of each worker in Regions 1, 2, 3, and 5.

For both fully and partially cross-trained teams, the system converges to an asymptotic behavior (fixed point or period-2 orbit). For each item produced, each worker spends some time working on the item and, possibly, some time idling (due to blocking or halting). Define W_i^{F} and D_i^{F} as the long-run average work time and idle time, respectively, per item produced of worker i in the fully cross-trained team, for $i = 1, 2$. Similarly, define W_i^{P} and D_i^{P} as the long-run average work time and idle time, respectively, per item produced of worker i in the partially cross-trained team, for $i = 1, 2$.

Recall that α_i is the long-run average portion of work content of each item covered by worker i. For the fullycross-trained team $\alpha_i = v_i W_i^{\mathrm{F}}$ and $\mathcal{T}^{\mathrm{F}} = 1/(W_i^{\mathrm{F}} + D_i^{\mathrm{F}})$. Let R_i^{F} denote the remuneration rate of worker i in the fully cross-trained team, for $i = 1, 2$. By definition, $R_i^{\mathrm{F}} = \alpha_i \mathcal{T}^{\mathrm{F}} = v_i W_i^{\mathrm{F}} / (W_i^{\mathrm{F}} + D_i^{\mathrm{F}})$. Let R_i^{P} denote the remuneration rate of worker i in the partiallycross-trained team, for $i = 1, 2$. Similarly, we have $R_i^{\mathrm{P}} = v_i W_i^{\mathrm{P}} / (W_i^{\mathrm{P}} + D_i^{\mathrm{P}})$.

We compare the remuneration rates of each worker for different teams in Regions 1, 2, 3, and 5. We analyze the remuneration rates of workers 1 and 2 separately as follows.

Worker 1: According to Theorem 2, in Regions 1, 2, 3, and 5 worker 1 is never idle after a transient periodin the partially cross-trained team. This implies $D_1^{\mathrm{P}} = 0$ and thus, $R_1^{\mathrm{P}} = v_1$, the maximum remuneration rate possible for worker 1.

According to Theorem 1, in Regions 1 and 2 worker 1 is constantly blocked at some locations in the fully cross-trained team. Thus, $D_1^{\mathrm{F}} > 0$. This implies $R_1^{\mathrm{F}} < v_1 = R_1^{\mathrm{P}}$. In Regions 3 and 5 worker 1 is never idle after a transient period and so, $R_1^{\mathrm{F}} = v_1 = R_1^{\mathrm{P}}$.

Worker 2: According to Theorem 2, in Regions 1, 2, 3, and 5 worker 2 repeatedly works only on station 2 in the partially cross-trained team and so, $W_2^{\mathrm{P}} = s_2/v_2$. For the fully cross-trained team, worker 2 covers at least s_2 for each item and so, $W_2^{\mathrm{F}} \geq s_2/v_2 = W_2^{\mathrm{P}}$ and the equality holds if and only if $s_3 = \frac{r}{r+1} - s_1$.

Now, we want to show that $D_2^{\mathrm{F}} \leq D_2^{\mathrm{P}}$. This requires some new notation. For the fully cross-trained team, worker 2 can be blocked at location $s_1 + s_2$ and can be halted at location 1 for each item produced. The total idling time per item produced is D_2^{F}. Let y_j^{F} denote the average work content on station j that worker 1 covers per item produced when worker 2 is idle (for the total idling time D_2^{F}), for $j = 1$ and 3. For the partially cross-trained team, worker 2 can only be idle at location $s_1 + s_2$ for each item produced. The idling time per item produced is D_2^{P}. Let y_j^{P} denote the average work content on station j that worker 1 covers per item produced during the idling time D_2^{P}, for $j = 1$ and 3.

We first show that $y_3^{\mathrm{F}} \leq y_3^{\mathrm{P}}$. We only need to consider the case where $y_3^{\mathrm{F}} > 0$. Note that for the fully cross-trained team, $y_3^{\mathrm{F}} > 0$ if and only if worker 2 is blocked at location $s_1 + s_2$, which is only possible in Region 3 according to Theorem 1. In Region 3, after the system converges to the fixed point, worker 1 remains in station 3 when worker 2 reaches location $s_1 + s_2$. Let β^{F} denote the horizontal position of worker 1 when worker 2 reaches location $s_1 + s_2$ for the fully cross-trained team. Similarly, for the partially cross-trained team, worker 1 remains in station 3 when worker 2 reaches location $s_1 + s_2$ in Region 3. Let β^{P} denote the horizontal position of worker 1 when worker 2 reaches location $s_1 + s_2$ for the partially cross-trained team. Since the hand-off position is less than or equal to s_1 for the fully cross-trained team, we have $\beta^{\mathrm{F}} \leq \beta^{\mathrm{P}}$. This implies $y_3^{\mathrm{F}} \leq y_3^{\mathrm{P}}$.

We then show that $y_1^{\mathrm{F}} \leq y_1^{\mathrm{P}}$. We only need to consider the case where $y_1^{\mathrm{F}} > 0$. Note that for the fully cross-trained team, $y_1^{\mathrm{F}} > 0$ if and only if worker 2 is halted at location 1, which is only possible in Region 1 according to Theorem 1. In Region 1, after the system converges to the fixed point, worker 1 works on station 1 when worker 2 reaches location

1. Let γ^{F} denote the horizontal position of worker 1 when worker 2 reaches location 1 for the fully cross-trained team. On the other hand, for the partially cross-trained team worker 1 can be in station 3 or 1 when worker 2 reaches location $s_1 + s_2$. If worker 1 is in station 3, then $y_1^{\mathrm{P}} = s_1 \geq y_1^{\mathrm{F}}$. Otherwise, let γ^{P} denote the horizontal position of worker 1 when worker 2 reaches location $s_1 + s_2$ for the partially cross-trained team. Since in Region 1 worker 1 is constantly blocked at location 0 for the fully cross-trained team, we have $\gamma^{\mathrm{F}} \geq \gamma^{\mathrm{P}}$. This implies $y_1^{\mathrm{F}} \leq y_1^{\mathrm{P}}$.

As a result, we have $y_1^{\mathrm{F}} + y_3^{\mathrm{F}} \leq y_1^{\mathrm{P}} + y_3^{\mathrm{P}}$. This implies $D_2^{\mathrm{F}} = (y_1^{\mathrm{F}} + y_3^{\mathrm{F}})/v_1 \leq (y_1^{\mathrm{P}} + y_3^{\mathrm{P}})/v_1 = D_2^{\mathrm{P}}$. Since $W_2^{\mathrm{F}} \geq W_2^{\mathrm{P}}$ and $D_2^{\mathrm{F}} \leq D_2^{\mathrm{P}}$, we have $R_2^{\mathrm{F}} = v_2 W_2^{\mathrm{F}}/(W_2^{\mathrm{F}} + D_2^{\mathrm{F}}) \geq v_2 W_2^{\mathrm{P}}/(W_2^{\mathrm{P}} + D_2^{\mathrm{P}}) = R_2^{\mathrm{P}}$ and the equality holds if and only if $s_3 = \frac{r}{r+1} - s_1$.

$\square$

A.2 Technical details for the three-station case in Chapter 4

A.2.1 Constructing the function f

To study the dynamics of the three-station, two-worker system, we first construct the function f. Figure A.4 shows five work-content regions. Each region corresponds to a

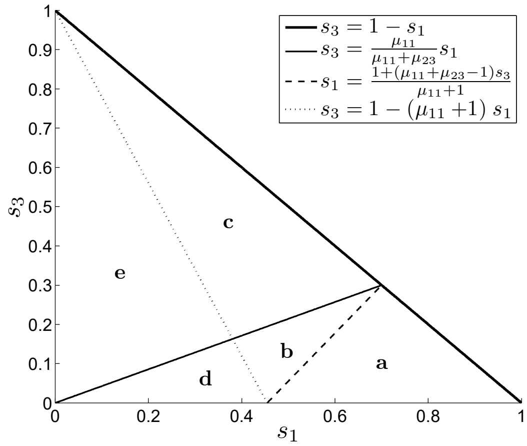

Figure A.4: **Five distinct forms of the function f.** Each region corresponds to a distinct form of the function f. We set $\mu_{11} = 1.2$, $\mu_{13} = 0.8$, $\mu_{21} = 0.7$, and $\mu_{23} = 1.6$ in this example.

distinct form of the function f, which is determined in Lemma 4. Let

$$\begin{aligned}
\theta_1 &= s_1 - \frac{\mu_{13}}{\mu_{13}+\mu_{21}} s_3; \\
\theta_2 &= \frac{1+(\mu_{13}+\mu_{21}-\mu_{11}-1)s_1 - (\mu_{13}+1)s_3}{\mu_{13}+\mu_{21}}; \\
\theta_3 &= \frac{1+(\mu_{13}+\mu_{21}-\mu_{11}-1)s_1 + (\mu_{11}+\mu_{23}-\mu_{13}-1)s_3}{\mu_{13}+\mu_{21}}; \\
\theta_4 &= \frac{1+(\mu_{13}+\mu_{21}-1)s_1 - (\mu_{13}+1)s_3}{\mu_{13}+\mu_{21}}.
\end{aligned}$$

Lemma 4. *The function f is given as follows.*

Regiona $\left(s_1 > \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}\right)$*:*

$$f(x_n) = s_1 - s_3.$$

Regionb $\left(s_3 \geq 1-(\mu_{11}+1)s_1,\ s_1 \leq \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1},\ \textit{and}\ s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}} s_1\right)$*:*

$$f(x_n) = \begin{cases} \eta_1, & \textit{if } x_n \in [s_1 - s_3,\ \theta_1); \\ \gamma(x_n), & \textit{if } x_n \in [\theta_1,\ \theta_3]; \\ s_1 - s_3, & \textit{otherwise.} \end{cases}$$

Regionc $\left(s_3 \geq 1-(\mu_{11}+1)s_1 \text{ and } s_3 \geq \frac{\mu_{11}}{\mu_{11}+\mu_{23}}s_1\right)$:

$$f(x_n) = \begin{cases} \eta_1, & \text{if } x_n \in [\max\{0, s_1-s_3\},\ \theta_1); \\ \gamma(x_n), & \text{if } x_n \in [\theta_1,\ \theta_4]; \\ \eta_2, & \text{otherwise.} \end{cases}$$

Regiond $\left(s_3 < 1-(\mu_{11}+1)s_1 \text{ and } s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}}s_1\right)$:

$$f(x_n) = \begin{cases} s_1, & \text{if } x_n \in [s_1-s_3,\ \theta_2); \\ \gamma(x_n), & \text{if } x_n \in [\theta_2,\ \theta_3]; \\ s_1-s_3, & \text{otherwise.} \end{cases}$$

Regione $\left(s_3 < 1-(\mu_{11}+1)s_1 \text{ and } s_3 \geq \frac{\mu_{11}}{\mu_{11}+\mu_{23}}s_1\right)$:

$$f(x_n) = \begin{cases} s_1, & \text{if } x_n \in [\max\{0, s_1-s_3\},\ \theta_2); \\ \gamma(x_n), & \text{if } x_n \in [\theta_2,\ \theta_4]; \\ \eta_2, & \text{otherwise.} \end{cases}$$

Proof. We construct the function f for the following two cases separately: (I) $s_1 > s_3$ and (II) $s_1 \leq s_3$.

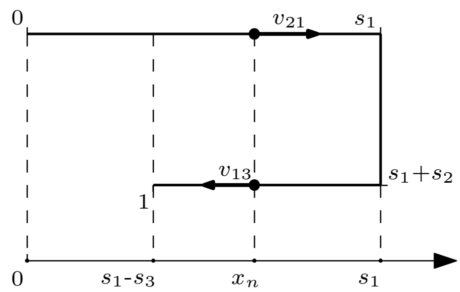

Figure A.5: **Case (I) ($s_1 > s_3$).** The hand-offposition x_n falls in the interval $[s_1 - s_3, s_1]$ The actual locations of workers on the line immediately after the n-th hand-off are shown.

For Case (I), the hand-off position x_n falls in the interval $[s_1 - s_3, s_1]$ on the horizontal axis. Figure A.5 shows the conceptual line for Case (I). Note that the actual locations of workers on the line immediately *after* the n-th hand-off are shown in the figure. In this case, W_1 may be blocked at location 0 or halted at location s_1 on the line, and W_2 may be blocked at location $s_1 + s_2$ or halted at location 1. We determine the next hand-off position x_{n+1} by considering all possible combinations of blocking and halting events.

(I) $s_1 > s_3$ ($x_n \in [s_1 - s_3, s_1]$):

(A) W_1 is not blocked at location 0 if $\frac{s_1-x_n}{v_{21}} \le \frac{x_n-s_1+s_3}{v_{13}} \Leftrightarrow x_n \ge \theta_1$.

(1) W_1 is not halted at location s_1 if $\frac{s_1-x_n}{v_{21}} + \frac{1-s_1-s_3}{v_{22}} \le \frac{x_n-s_1+s_3}{v_{13}} + \frac{s_1}{v_{11}} \Leftrightarrow x_n \ge \theta_2$.

(a) W_2 is not blocked at location $s_1 + s_2$ if $\frac{s_1-x_n}{v_{21}} + \frac{1-s_1-s_3}{v_{22}} \ge \frac{x_n-s_1+s_3}{v_{13}} \Leftrightarrow$ $x_n \le \theta_4$.

(i) W_2 is not halted at location 1 if

$\frac{s_1-x_n}{v_{21}} + \frac{1-s_1-s_3}{v_{22}} + \frac{s_3}{v_{23}} \ge \frac{x_n-s_1+s_3}{v_{13}} + \frac{s_1-s_3}{v_{11}} \Leftrightarrow x_n \le \theta_3$.

— In this case, $x_{n+1} = \gamma(x_n)$.

(ii) W_2 is halted at location 1 if $x_n > \theta_3$.

— In this case, $x_{n+1} = s_1 - s_3$.

(b) W_2 is blocked at location $s_1 + s_2$ if $x_n > \theta_4$.

(i) W_2 is not halted at location 1 if $\frac{s_3}{v_{23}} \ge \frac{s_1-s_3}{v_{11}} \Leftrightarrow s_3 \ge \frac{\mu_{11}}{\mu_{11}+\mu_{23}} s_1$.

— In this case, $x_{n+1} = \eta_2$.

(ii) W_2 is halted at location 1 if $s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}} s_1$.

— In this case, $x_{n+1} = s_1 - s_3$.

(2) W_1 is halted at location s_1 if $x_n < \theta_2$.

— In this case, $x_{n+1} = s_1$.

(B) W_1 is blocked at location 0 if $x_n < \theta_1$.

(1) W_1 is not halted at location s_1 if $\frac{1-s_1-s_3}{v_{22}} \leq \frac{s_1}{v_{11}} \Leftrightarrow s_3 \geq 1-(\mu_{11}+1)s_1$.

(a) W_2 is not halted at location 1 if $\frac{1-s_1-s_3}{v_{22}} + \frac{s_3}{v_{23}} \geq \frac{s_1-s_3}{v_{11}} \Leftrightarrow s_1 \leq \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}$.

— In this case, $x_{n+1} = \eta_1$.

(b) W_2 is halted at location 1 if $s_1 > \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}$.

— In this case, $x_{n+1} = s_1 - s_3$.

(2) W_1 is halted at location s_1 if $s_3 < 1-(\mu_{11}+1)s_1$.

— In this case, $x_{n+1} = s_1$.

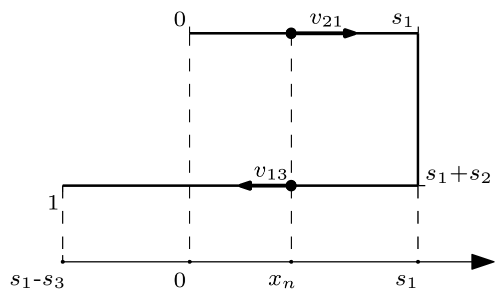

Figure A.6: **Case (II) ($s_1 \leq s_3$).** The hand-off position x_n falls in the interval $[0, s_1]$ on the horizontal axis. The actual locations of workers on the line immediately after the n-th hand-off are shown.

For Case (II), the hand-off position x_n falls in the interval $[0, s_1]$ on the horizontal axis. Figure A.6 shows the conceptual line for Case (II). The actual locations of workers on the line immediately after the n-th hand-off are shown in the figure. In this case, W_1 may be blocked at location 0 or halted at location s_1 on the line, and W_2 may be blocked at location $s_1 + s_2$. We determine the next hand-off position x_{n+1} by considering all possible combinations of blocking and halting events.

(II) $s_1 \leq s_3$ ($x_n \in [0, s_1]$):

(A) W_1 is not blocked at location 0 if $x_n \geq \theta_1$.

(1) W_1 is not halted at location s_1 if $x_n \geq \theta_2$.

(a) W_2 is not blocked at location $s_1 + s_2$ if $x_n \leq \theta_4$.

— In this case, $x_{n+1} = \gamma(x_n)$.

(b) W_2 is blocked at location $s_1 + s_2$ if $x_n > \theta_4$.

— In this case, $x_{n+1} = \eta_2$.

(2) W_1 is halted at location s_1 if $x_n < \theta_2$.

— In this case, $x_{n+1} = s_1$.

(B) W_1 is blocked at location 0 if $x_n < \theta_1$.

(1) W_1 is not halted at location s_1 if $s_3 \geq 1 - (\mu_{11} + 1)s_1$.

— In this case, $x_{n+1} = \eta_1$.

(2) W_1 is halted at location s_1 if $s_3 < 1 - (\mu_{11} + 1)s_1$.

— In this case, $x_{n+1} = s_1$.

Now, we check the function f in each region of Figure A.4 using the above results. Note that $\theta_3 > \theta_2$, $\theta_4 > \theta_1$, and $\theta_4 > \theta_2$.

Region a: In this region, we have $s_1 > \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}$, which implies $\theta_1 > \theta_3 > \theta_2$, $s_3 > 1-(\mu_{11}+1)s_1$, and $s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}}s_1$. (The last inequality is implied by $s_1+s_3 < 1$: The lines $s_1 = \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}$ and $s_3 = \frac{\mu_{11}}{\mu_{11}+\mu_{23}}s_1$ always intersect at point $(\frac{\mu_{11}+\mu_{23}}{2\mu_{11}+\mu_{23}}, \frac{\mu_{11}}{2\mu_{11}+\mu_{23}})$ on the line $s_1 + s_3 = 1$. See Figure A.4.)

Since $s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}}s_1$, we have $s_1 > s_3$. Thus, this region corresponds to Case (I). If $x_n < \theta_1$ then, because of inequalities $s_3 > 1-(\mu_{11}+1)s_1$ and $s_1 > \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}$, we have

Case (I)(B)(1)(b): $x_{n+1} = s_1 - s_3$. Otherwise, we have $x_n \geq \theta_1 > \theta_3 > \theta_2$ and so the region corresponds to Case (I)(A)(1). In addition, we have $s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}} s_1 \Leftrightarrow \theta_4 > \theta_3$. Thus, we have either Case (I)(A)(1)(a)(ii) due to the inequality $x_n > \theta_3$ or Case (I)(A)(1)(b)(ii) due to the inequality $s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}} s_1$. Both cases imply $x_{n+1} = s_1 - s_3$. Therefore, for any x_n, we have $x_{n+1} = s_1 - s_3$ in this region.

Region b: In this region, we have $s_3 \geq 1 - (\mu_{11} + 1)s_1$, $s_1 \leq \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}$, and $s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}} s_1$. The last inequality implies $s_1 > s_3$, and thus this region corresponds to Case (I). If $x_n < \theta_1$ then, because of the inequalities $s_3 \geq 1 - (\mu_{11} + 1)s_1$ and $s_1 \leq \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}$, we have Case (I)(B)(1)(a): $x_{n+1} = \eta_1$. Otherwise, we have $x_n \geq \theta_1$. Since $s_3 \geq 1 - (\mu_{11} + 1)s_1 \Leftrightarrow \theta_1 \geq \theta_2$, this region corresponds to Case (I)(A)(1). In addition, $s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}} s_1 \Leftrightarrow \theta_4 > \theta_3$. Thus, if $x_n \leq \theta_3 < \theta_4$, we have Case (I)(A)(1)(a)(i): $x_{n+1} = \gamma(x_n)$. Otherwise, we have $x_n > \theta_3$, and thus we have either Case (I)(A)(1)(a)(ii) or Case (I)(A)(1)(b)(ii) due to the inequality $s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}} s_1$. Both cases imply $x_{n+1} = s_1 - s_3$.

Region c: In this region, we have $s_3 \geq 1-(\mu_{11}+1)s_1$ and $s_3 \geq \frac{\mu_{11}}{\mu_{11}+\mu_{23}} s_1$, which imply $\theta_1 \geq \theta_2$ and $\theta_3 \geq \theta_4$ respectively. In addition, as shown in Region a, if $s_1 > \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}$, then $s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}} s_1$. Thus, in this region we have $s_3 \geq \frac{\mu_{11}}{\mu_{11}+\mu_{23}} s_1 \Rightarrow s_1 \leq \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}$. Both Cases (I) and (II) are possible in this region.

For Case (I), if $x_n < \theta_1$ then, because of the inequalities $s_3 \geq 1 - (\mu_{11} + 1)s_1$ and $s_1 \leq \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}$, we have Case (I)(B)(1)(a): $x_{n+1} = \eta_1$. Otherwise, we have $x_n \geq \theta_1 \geq \theta_2$, and thus this region corresponds to Case (I)(A)(1). If $x_n \leq \theta_4$ then, because of the inequality $\theta_3 \geq \theta_4$, we have Case (I)(A)(1)(a)(i): $x_{n+1} = \gamma(x_n)$. Otherwise, we have $x_n > \theta_4$. Since $s_3 \geq \frac{\mu_{11}}{\mu_{11}+\mu_{23}} s_1$, we have Case (I)(A)(1)(b)(i): $x_{n+1} = \eta_2$.

For Case (II), if $x_n < \theta_1$ then, because of the inequality $s_3 \geq 1 - (\mu_{11} + 1)s_1$, we have Case (II)(B)(1): $x_{n+1} = \eta_1$. Otherwise, we have $x_n \geq \theta_1 \geq \theta_2$, and thus this region corresponds to Case (II)(A)(1). If $x_n \leq \theta_4$, thenwe have Case (II)(A)(1)(a): $x_{n+1} = \gamma(x_n)$. Otherwise, we have $x_n > \theta_4$, and thus we have Case (II)(A)(1)(b): $x_{n+1} = \eta_2$.

Region d: In this region, we have $s_3 < 1 - (\mu_{11} + 1)s_1$ and $s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}} s_1$. The first inequality implies $\theta_2 > \theta_1$, and the second inequality implies $s_1 > s_3$ and $\theta_4 > \theta_3$. Thus, this region corresponds to Case (I). If $x_n < \theta_2$, then we have either Case (I)(B)(2) due to the inequality $s_3 < 1 - (\mu_{11} + 1)s_1$ or Case (I)(A)(2). Both cases imply $x_{n+1} = s_1$. If $\theta_2 \leq x_n \leq \theta_3$, then we have Case (I)(A)(1)(a)(i): $x_{n+1} = \gamma(x_n)$. Otherwise, we have $x_n > \theta_3$, and thus we have either Case (I)(A)(1)(a)(ii) or Case (I)(A)(1)(b)(ii) due to the inequality $s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}} s_1$. Both cases imply $x_{n+1} = s_1 - s_3$.

Region e: In this region, we have $s_3 < 1 - (\mu_{11} + 1)s_1$ and $s_3 \geq \frac{\mu_{11}}{\mu_{11}+\mu_{23}} s_1$, which imply $\theta_2 > \theta_1$ and $\theta_3 \geq \theta_4$ respectively. Both Cases (I) and (II) are possible in this region.

For Case (I), if $x_n < \theta_2$, then we have either Case (I)(B)(2) due to the inequality $s_3 < 1 - (\mu_{11} + 1)s_1$ or Case (I)(A)(2). Both cases imply $x_{n+1} = s_1$. If $\theta_2 \leq x_n \leq \theta_4$ then, because of the inequality $\theta_3 \geq \theta_4$, we have Case (I)(A)(1)(a)(i): $x_{n+1} = \gamma(x_n)$. Otherwise, we have $x_n > \theta_4$. Since $s_3 \geq \frac{\mu_{11}}{\mu_{11}+\mu_{23}} s_1$, we have Case (I)(A)(1)(b)(i): $x_{n+1} = \eta_2$.

For Case (II), if $x_n < \theta_2$, then we have either Case (II)(B)(2) due toinequality $s_3 < 1 - (\mu_{11} + 1)s_1$ or Case (II)(A)(2). Both cases imply $x_{n+1} = s_1$. If $\theta_2 \leq x_n \leq \theta_4$, then we have Case (II)(A)(1)(a): $x_{n+1} = \gamma(x_n)$. Otherwise, we have $x_n > \theta_4$, and thus we have Case (II)(A)(1)(b): $x_{n+1} = \eta_2$. $\square$

A.2.2 Dynamics of a piecewise-linear function

We need the following lemma to determine the asymptotic behaviors of the three-station U-line.

Lemma 5. *For any $\rho > 0$, suppose $x_{n+1} = g(x_n)$ and $g : [A, B] \mapsto [A, B]$ $(0 \leq A < B)$ has the following form*

$$g(x) \quad = \quad \begin{cases} Y, & \textit{if } x \in [A, X_1); \\ Y + \rho X_1 - \rho x, & \textit{if } x \in [X_1, X_2]; \\ Y + \rho X_1 - \rho X_2, & \textit{otherwise}; \end{cases}$$

where Y, X_1, and X_2 are constants. The asymptotic behaviors of the system can be summarized as follows.

(I) $Y \leq X_1$**:** *The system converges to a fixed point Y.*

(II) $X_1 < Y < (1+\rho)X_2 - \rho X_1$**:** *There are three cases:*

- **(1)** $\rho < 1$**:** *The system converges to a fixed point $\frac{Y+\rho X_1}{1+\rho}$;*
- **(2)** $\rho = 1$**:** *The system converges to a period-2 orbit: x and $Y + \rho X_1 - \rho x$, where x depends on the initial point of the orbit;*
- **(3)** $\rho > 1$**:** *The system converges to a period-2 orbit:*
  - **a.** $X_1 < Y \leq X_2$**:** *Period-2 orbit: Y and $(1-\rho)Y + \rho X_1$;*
  - **b.** $X_2 < Y < X_2 + (\rho-1)(X_2 - X_1)$**:** *Period-2 orbit: Y and $Y + \rho X_1 - \rho X_2$;*
  - **c.** $X_2 + (\rho-1)(X_2 - X_1) \leq Y < (1+\rho)X_2 - \rho X_1$**:** *Period-2 orbit: $Y + \rho X_1 - \rho X_2$ and $\rho^2 X_2 - (\rho-1)Y - \rho(\rho-1)X_1$.*

(III) $Y \geq (1+\rho)X_2 - \rho X_1$**:** *The system converges to a fixed point* $Y + \rho X_1 - \rho X_2$.

Proof. We first prove case (I). If $Y \leq X_1$, then for any initial point $x \in [A, B]$ we have $g(x) \leq Y \leq X_1$. Thus, $g(g(x)) = Y$ and the system stays at the fixed point Y. Similarly, we can prove case (III) as follows. If $Y \geq (1+\rho)X_2 - \rho X_1$, then for any initial point $x \in [A, B]$ we have $g(x) \geq Y + \rho X_1 - \rho X_2 \geq X_2$. Thus, $g(g(x)) = Y + \rho X_1 - \rho X_2$ and the system stays at the fixed point $Y + \rho X_1 - \rho X_2$.

Now, we prove case (II). If $X_1 < Y < (\rho+1)X_2 - \rho X_1$, there are three possible cases: (1) $\rho < 1$, (2) $\rho = 1$, and (3) $\rho > 1$. We analyze each case as follows.

For case (1), we have $\rho < 1$. For any initial point $x_0 \in [A, B]$, we have $|f(x_n) - \eta_0| \leq \rho^n |x_0 - \eta_0|$. Since $\rho < 1$, the system converges to the fixed point η_0.

For case (2), we have $\rho = 1$. For any initial point $x_0 \in [A, B]$, there are three possible cases:

a. If $x_0 < X_1$, then $g(x_0) = Y$.

- If $X_1 < Y \leq X_2$, then $g(Y) = Y + X_1 - Y = X_1$. Thus, $g(g(Y)) = Y$, which means the system converges to a period-2 orbit: Y and X_1.
- If $X_2 < Y < 2X_2 - X_1$, then $g(Y) = Y + X_1 - X_2$. We have $g(Y) \in [X_1, X_2]$. Thus, $g(g(Y)) = Y + X_1 - g(Y) = X_2$, which implies $g(g(g(Y))) = Y + X_1 - X_2 = g(Y)$. The system converges to a period-2 orbit: $Y + X_1 - X_2$ and X_2.

b. If $x_0 > X_2$, then $g(x_0) = Y + X_1 - X_2$. Let $Y' = Y + X_1 - X_2$. We have $2X_1 - X_2 < Y' < X_2$.

- If $2X_1 - X_2 < Y' < X_1$, then $g(Y') = Y$. We have $g(Y') \in [X_1, X_2]$. Thus, $g(g(Y')) = Y + X_1 - g(Y') = X_1$, which implies $g(g(g(Y'))) = Y + X_1 - X_1 =$

$Y = g(Y')$. The system converges to a period-2 orbit: Y and X_1.

- If $X_1 \leq Y' < X_2$, then $g(Y') = Y + X_1 - Y' = X_2$. Thus, $g(g(Y')) = Y + X_1 - X_2 = Y'$. The system converges to a period-2 orbit: Y' and X_2.

c. If $X_1 \leq x_0 \leq X_2$, then $g(x_0) = Y + X_1 - x_0$.

- If $X_1 \leq g(x_0) \leq X_2$, then $g(g(x_0)) = x_0$. The system converges to a period-2 orbit: x_0 and $Y + X_1 - x_0$.
- If $g(x_0) < X_1$, then $g(g(x_0)) = Y$ and this reduces to case a.
- If $g(x_0) > X_2$, then $g(g(x_0)) = Y + X_1 - X_2$ and this reduces to case b.

For case (3), we have $\rho > 1$. We first prove by contradictionthat for any initial point $x_0 \in [A, B]$, such that $x_0 \neq \eta_0$, the orbit under g contains at least one endpoint Y or $Y + \rho X_1 - \rho X_2$. If not, then $x_n \in (X_1, X_2)$ for all $n = 0, 1, 2, \ldots$. However, since $|x_n - \eta_0| = \rho^n |x_0 - \eta_0|$, there exists a n' such that $x_{n'} \notin (X_1, X_2)$. This contradicts our assumption. Thus, any orbit under g contains at least one endpoint Y or $Y + \rho X_1 - \rho X_2$. As a result, we can focus our analysis on orbits starting from Y or $Y + \rho X_1 - \rho X_2$. There are three cases:

a. If $X_1 < Y \leq X_2$, then $Y < X_2 + (\rho - 1)(X_2 - X_1) \Leftrightarrow Y + \rho X_1 - \rho X_2 < X_1$, which implies $g(Y + \rho X_1 - \rho X_2) = Y$. Thus, we only need to analyze orbits starting from Y. $g(Y) = (1 - \rho)Y + \rho X_1 < X_1$, which implies $g(g(Y)) = Y$. Therefore, the system converges to a period-2 orbit: Y and $(1 - \rho)Y + \rho X_1$.

b. If $X_2 < Y < X_2 + (\rho - 1)(X_2 - X_1)$, then $g(Y) = Y + \rho X_1 - \rho X_2$ and $g(Y + \rho X_1 - \rho X_2) = Y$. Thus, the system converges to a period-2 orbit: Y and $Y + \rho X_1 - \rho X_2$.

c. If $X_2 + (\rho - 1)(X_2 - X_1) \leq Y < (\rho + 1)X_2 - \rho X_1$, then $Y > X_2$, which implies $g(Y) = Y + \rho X_1 - \rho X_2$. Thus, we only need to analyze orbits starting from $Y + \rho X_1 - \rho X_2$. The inequalities $X_2 + (\rho - 1)(X_2 - X_1) \leq Y < (\rho + 1)X_2 - \rho X_1$ imply $X_1 \leq Y + \rho X_1 - \rho X_2 < X_2$, and so $g(Y + \rho X_1 - \rho X_2) = Y + \rho X_1 - \rho(Y + \rho X_1 - \rho X_2) = \rho^2 X_2 - (\rho - 1)Y - \rho(\rho - 1)X_1 > X_2$. The last inequality is implied by $Y < (\rho+1)X_2 - \rho X_1$. Thus, $g(g(Y + \rho X_1 - \rho X_2)) = Y + \rho X_1 - \rho X_2$. Therefore, the system converges to a period-2 orbit: $Y + \rho X_1 - \rho X_2$ and $\rho^2 X_2 - (\rho - 1)Y - \rho(\rho - 1)X_1$.

□

Note that the function g in Lemma 5 represents a general form of the function f in Lemma 4. In Appendix A.2, we use the properties of the function g described in Lemma 5 to determine the asymptotic behaviors of the three-station, two-worker system.

A.2.3 Asymptotic behaviors andthroughput

Lemma 6. *If $\varphi \leq 1$, the two-worker cellular bucket brigade on a three-station u-line has a distinct asymptotic behavior in each of the following five regions.*

Region 1: *This region is defined by $s_1 > \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}$. The system converges to a fixed point $x^* = s_1 - s_3$. At the fixed point, W_1 is constantly blocked at location 0 and W_2 is constantly halted at location 1. The average throughput is $\mathcal{T} = \left(\frac{s_1-s_3}{v_{11}} + \frac{s_3}{v_{21}}\right)^{-1}$.*

Region 2: *This region is defined by $s_1 < \frac{1+(\mu_{11}+\mu_{23}-1)s_3}{\mu_{11}+1}$ and $s_1 > \frac{1}{\mu_{11}+1} + \frac{\mu_{11}\mu_{13}+\mu_{13}\mu_{23}-\mu_{13}-\mu_{21}}{(\mu_{11}+1)(\mu_{13}+\mu_{21})} \cdot s_3$. The system converges to a fixed point $x^* = \eta_1$. At the fixed point, W_1 is constantly blocked at location 0. The average throughput is $\mathcal{T} = \left(\frac{\eta_1}{v_{11}} + \frac{s_1-\eta_1}{v_{21}}\right)^{-1}$.*

Region 3: *This region is defined by* $s_3 > \frac{1}{\mu_{13}+1} + \frac{\mu_{11}\mu_{13}+\mu_{11}\mu_{21}-\mu_{11}-\mu_{23}}{(\mu_{13}+1)(\mu_{11}+\mu_{23})} \cdot s_1$*. The system converges to a fixed point* $x^* = \eta_2$*. At the fixed point,* W_2 *is constantly blocked at location* $s_1 + s_2$*. The average throughput is* $\mathcal{T} = \left(\frac{\eta_2}{v_{11}} + \frac{s_3-s_1+\eta_2}{v_{13}}\right)^{-1}$*.*

Region 4: *This region is defined by* $s_3 < \frac{1-(\mu_{11}+1)s_1}{\mu_{13}+1}$*. The system converges to a fixed point* $x^* = s_1$*. At the fixed point,* W_1 *is constantly halted at location* s_1*. The average throughput is* $\mathcal{T} = \left(\frac{1-s_1-s_3}{v_{22}}\right)^{-1}$*.*

Region 5: *This region is defined by* $s_1 < \frac{1}{\mu_{11}+1} + \frac{\mu_{11}\mu_{13}+\mu_{13}\mu_{23}-\mu_{13}-\mu_{21}}{(\mu_{11}+1)(\mu_{13}+\mu_{21})} \cdot s_3$*,* $s_3 < \frac{1}{\mu_{13}+1} + \frac{\mu_{11}\mu_{13}+\mu_{11}\mu_{21}-\mu_{11}-\mu_{23}}{(\mu_{13}+1)(\mu_{11}+\mu_{23})} \cdot s_1$*, and* $s_3 > \frac{1-(\mu_{11}+1)s_1}{\mu_{13}+1}$*. If* $\varphi < 1$*, the system converges to a fixed point* η_0*. If* $\varphi = 1$*, the system converges to a period-2 orbit:* x *and* $\gamma(x)$*, where* x *depends on the initial locations of the workers. Neither blocking nor halting occurs in this region. The average throughput is* $\mathcal{T} = \left(\frac{\eta_0}{v_{11}} + \frac{s_3-s_1+\eta_0}{v_{13}}\right)^{-1}$*.*

Proof. We partition the entire feasible work-content area into five regions shown in Figure 4.4(a). We determine the asymptotic behavior and throughput of the system in each region separately.

Region 1: This region is identical to Region a in Figure A.4. Lemma 4 shows that the system always converges to the fixed point $x^* = s_1 - s_3$. When the system operates on the fixed point, according to the proof of Lemma 4 (see Region a), W_1 is constantly blocked at location 0 and W_2 is constantly halted at location 1. The average throughput is $\mathcal{T} = \left(\frac{s_3}{v_{21}} + \frac{s_1-s_3}{v_{11}}\right)^{-1}$.

Region 2: This region falls in Regions b and c in Figure A.4. Since $s_1 > \frac{1}{\mu_{11}+1} + \frac{\mu_{11}\mu_{13}+\mu_{13}\mu_{23}-\mu_{13}-\mu_{21}}{(\mu_{11}+1)(\mu_{13}+\mu_{21})} \cdot s_3 \Leftrightarrow \eta_1 < \theta_1$, according to the function f in Regions b and c as well as Lemma 5, the system converges to the fixed point $x^* = \eta_1$. When the system operates on the fixed point, according to the proof of Lemma 4 (see Regions b and c), W_1 is constantly blocked at location 0. The average throughput is $\mathcal{T} = \left(\frac{s_1-\eta_1}{v_{21}} + \frac{\eta_1}{v_{11}}\right)^{-1}$.

Region 3: This region falls in Regions c and e in Figure A.4. Since $s_3 > \frac{1}{\mu_{13}+1} + \frac{\mu_{11}\mu_{13}+\mu_{11}\mu_{21}-\mu_{11}-\mu_{23}}{(\mu_{13}+1)(\mu_{11}+\mu_{23})} \cdot s_1 \Leftrightarrow \eta_2 > \theta_4$, according to the function f in Regions c and e as well as Lemma 5, the system converges to the fixed point $x^* = \eta_2$. When the system operates on the fixed point, according to the proof of Lemma 4 (see Regions c and e), W_2 is constantly blocked at location $s_1 + s_2$. The average throughput is $\mathcal{T} = \left(\frac{s_3-s_1+\eta_2}{v_{13}} + \frac{\eta_2}{v_{11}}\right)^{-1}$.

Region 4: Since $s_3 < \frac{1-(\mu_{11}+1)s_1}{\mu_{13}+1} \Leftrightarrow \frac{s_3}{v_{13}} + \frac{s_1}{v_{11}} < \frac{1-s_1-s_3}{v_{22}}$, W_1 is constantly halted at location s_1. The system converges to the fixed point $x^* = s_1$. The average throughput is $\mathcal{T} = \left(\frac{1-s_1-s_3}{v_{22}}\right)^{-1}$.

Region 5: This region falls in Regions b, c, d, and e in Figure A.4. According to the function f in Regions b, c, d, and e as well as Lemma 5, we have (1) if $\varphi < 1$, the system converges to a fixed point η_0, and (2) if $\varphi = 1$, the system converges to a period-2 orbit: x and $\gamma(x)$, where x depends on the initial locations of the workers. Neither blocking nor halting occurs in this region. The average throughput is $\mathcal{T} = \left(\frac{\eta_0}{v_{11}} + \frac{s_3-s_1+\eta_0}{v_{13}}\right)^{-1}$. $\square$

According to the proof of Lemma 6, the asymptotic behaviors in Regions 1 to 4 are independent of φ. Thus, if $\varphi > 1$, the asymptotic behaviors and the expressions of the throughput remain the same in all regions except for Region 5.

Lemma 7. *If $\varphi > 1$, Region 5 can be partitioned into the following sevensubregions. Each subregion corresponds to a distinct period-2 orbit.*

Region 5a: *This subregion is defined by $s_3 > 1 - (\mu_{11} + 1)s_1$,*

$s_1 > \frac{1}{\mu_{11}+1} \cdot \left[1 + \left(\frac{(\mu_{11}+\mu_{23})(\mu_{13}-\mu_{11}-\mu_{23})}{\mu_{13}+\mu_{21}-\mu_{11}-\mu_{23}} - 1\right) s_3\right]$, *and*

$s_1 > \frac{\mu_{13}+\mu_{21}-\mu_{11}-\mu_{23}+(\mu_{11}\mu_{13}+\mu_{13}\mu_{23}+\mu_{11}+\mu_{23}-\mu_{13}-\mu_{21})s_3}{\mu_{11}\mu_{13}+\mu_{11}\mu_{21}+\mu_{13}+\mu_{21}-\mu_{11}-\mu_{23}}$. *The system converges to a period-2 orbit:* η_1 *and* $\gamma(\eta_1)$. *At the period-2 orbit,* W_1 *is blocked at location* 0 *for every other hand-off. The average throughput is*
$\mathcal{T} = \left[\left(s_1 - \frac{\eta_1+\gamma(\eta_1)}{2}\right)\left(\frac{1}{v_{21}} + \frac{1}{v_{23}}\right) + \frac{1-s_1-s_3}{v_{22}}\right]^{-1}$.

Region 5b: *This subregion is defined by* $s_3 > \frac{\mu_{11}}{\mu_{11}+\mu_{23}}s_1$, $s_3 > \frac{\mu_{11}(\mu_{13}+\mu_{21})}{\mu_{13}(\mu_{11}+\mu_{23})}s_1$, *and* $s_3 > \frac{1}{\mu_{13}+1} + \frac{\mu_{11}(\mu_{13}+\mu_{21})-(\mu_{11}+1)(\mu_{11}+\mu_{23})}{(\mu_{13}+1)(\mu_{11}+\mu_{23})}s_1$. *The system converges to a period-2 orbit:* η_2 *and* $\gamma(\eta_2)$. *At the period-2 orbit,* W_2 *is blocked at location* $s_1 + s_2$ *for every other hand-off. The average throughput is* $\mathcal{T} = \left(\frac{\eta_2+\gamma(\eta_2)}{2v_{11}} + \frac{2s_3-2s_1+\eta_2+\gamma(\eta_2)}{2v_{13}}\right)^{-1}$.

Region 5c: *This subregion is defined by* $s_3 < 1-(\mu_{11}+1)s_1$, $s_3 < \frac{1-s_1}{\mu_{13}+1}$, *and* $s_1 < \frac{1+(\mu_{11}+\mu_{23}-\mu_{13}-1)s_3}{\mu_{11}+1}$. *The system converges to a period-2 orbit:* s_1 *and* $\gamma(s_1)$. *At the period-2 orbit,* W_1 *is halted at location* s_1 *for every other hand-off. The average throughput is* $\mathcal{T} = \left[\frac{s_1-\gamma(s_1)}{2}\cdot\left(\frac{1}{v_{21}} + \frac{1}{v_{23}}\right) + \frac{1-s_1-s_3}{v_{22}}\right]^{-1}$.

Region 5d: *This subregion is defined by* $s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}}s_1$, $s_3 > 1-(\mu_{11}+1)s_1$, *and* $s_1 < \frac{1}{\mu_{11}+1}\cdot\left[1+\left(\frac{(\mu_{11}+\mu_{23})(\mu_{13}-\mu_{11}-\mu_{23})}{\mu_{13}+\mu_{21}-\mu_{11}-\mu_{23}} - 1\right)s_3\right]$. *The system converges to a period-2 orbit:* η_1 *and* $s_1 - s_3$. *At the period-2 orbit,* W_1 *is blocked at location* 0 *for every other hand-off, and* W_2 *is halted at location* 1 *for every other hand-off. The average throughput is* $\mathcal{T} = \left(\frac{s_1-s_3+\eta_1}{2v_{11}} + \frac{s_3-s_1+\eta_1}{2v_{13}} + \frac{s_3}{2v_{21}}\right)^{-1}$.

Region 5e: *This subregion is defined by* $s_3 > \frac{\mu_{11}}{\mu_{11}+\mu_{23}}s_1$, $s_3 > 1-(\mu_{11}+1)s_1$, $s_1 < \frac{\mu_{13}+\mu_{21}-\mu_{11}-\mu_{23}+(\mu_{11}\mu_{13}+\mu_{13}\mu_{23}+\mu_{11}+\mu_{23}-\mu_{13}-\mu_{21})s_3}{\mu_{11}\mu_{13}+\mu_{11}\mu_{21}+\mu_{13}+\mu_{21}-\mu_{11}-\mu_{23}}$, *and* $s_3 < \frac{\mu_{11}(\mu_{13}+\mu_{21})}{\mu_{13}(\mu_{11}+\mu_{23})}s_1$. *The system converges to a period-2 orbit:* η_1 *and* η_2. *At the period-2 orbit,* W_1 *is blocked at location* 0 *for every other hand-off, and* W_2 *is blocked at location* s_1+s_2 *for every other hand-off. The average throughput is* $\mathcal{T} = \left(\frac{\eta_1+\eta_2}{2v_{11}} + \frac{s_3-s_1+\eta_1}{2v_{13}} + \frac{s_1-\eta_2}{2v_{21}}\right)^{-1}$.

Region 5f: *This subregion is definedby* $s_3 > \frac{\mu_{11}}{\mu_{11}+\mu_{23}} s_1$, $s_3 < 1-(\mu_{11}+1)s_1$, $s_3 < \frac{1}{\mu_{13}+1} + \frac{\mu_{11}(\mu_{13}+\mu_{21})-(\mu_{11}+1)(\mu_{11}+\mu_{23})}{(\mu_{13}+1)(\mu_{11}+\mu_{23})} s_1$, *and* $s_3 > \frac{1-s_1}{\mu_{13}+1}$. *The system converges to a period-2 orbit:* s_1 *and* η_2. *At the period-2 orbit,* W_1 *is halted at location* s_1 *for every other hand-off, and* W_2 *is blocked at location* s_1+s_2 *for every other hand-off. The average throughput is* $\mathcal{T} = \left(\frac{\eta_2}{2v_{11}} + \frac{s_3}{2v_{13}} + \frac{s_1-\eta_2}{2v_{21}} + \frac{1-s_1-s_3}{2v_{22}}\right)^{-1}$.

Region 5g: *This subregion is defined by* $s_3 < \frac{\mu_{11}}{\mu_{11}+\mu_{23}} s_1$, $s_3 < 1-(\mu_{11}+1)s_1$, *and* $s_1 > \frac{1+(\mu_{11}+\mu_{23}-\mu_{13}-1)s_3}{\mu_{11}+1}$. *The system converges to a period-2 orbit:* s_1 *and* s_1-s_3. *At the period-2 orbit,* W_1 *is first blocked at location* 0 *and then halted at location* s_1 *for every other hand-off, and* W_2 *is halted at location* 1 *for every other hand-off. The average throughput is* $\mathcal{T} = \left(\frac{s_1-s_3}{2v_{11}} + \frac{s_3}{2v_{13}} + \frac{s_3}{2v_{21}} + \frac{1-s_1-s_3}{2v_{22}}\right)^{-1}$.

Proof. Region 5 falls in Regions b, c, d, and e in Figure A.4. We partition Region 5 into seven subregions shown in Figure 4.5(a). We determine the asymptotic behavior and throughput of the system in each subregion separately.

Region 5a: This subregion falls in Regions b and c in Figure A.4. According to the function f in Regions b and c as well as Lemma 5, the system converges to a period-2 orbit: η_1 and $\gamma(\eta_1)$. When the system operates on the period-2 orbit, according to the proof of Lemma 4 (see Regions b and c), W_1 is blocked at location 0 for every other hand-off. The average throughput is $\mathcal{T} = 2\left(\frac{s_1-\eta_1}{v_{21}} + \frac{1-s_1-s_3}{v_{22}} + \frac{s_1-\gamma(\eta_1)}{v_{23}} + \frac{s_1-\gamma(\eta_1)}{v_{21}} + \frac{1-s_1-s_3}{v_{22}} + \frac{s_1-\eta_1}{v_{23}}\right)^{-1} = \left[\left(s_1 - \frac{\eta_1+\gamma(\eta_1)}{2}\right)\left(\frac{1}{v_{21}} + \frac{1}{v_{23}}\right) + \frac{1-s_1-s_3}{v_{22}}\right]^{-1}$.

Region 5b: This subregion falls in Regions c and e in Figure A.4. According to the function f in Regions c and e as well as Lemma 5, the system converges to a period-2 orbit: η_2 and $\gamma(\eta_2)$. When the system operates on the period-2 orbit, according to the proof of Lemma 4 (see Regions c and e), W_2 is blocked at location s_1+s_2 for every other hand-off. The average throughput is $\mathcal{T} = 2\left(\frac{s_3-s_1+\eta_2}{v_{13}} + \frac{\gamma(\eta_2)}{v_{11}} + \frac{s_3-s_1+\gamma(\eta_2)}{v_{13}} + \frac{\eta_2}{v_{11}}\right)^{-1} = \left(\frac{\eta_2+\gamma(\eta_2)}{2v_{11}} + \frac{2s_3-2s_1+\eta_2+\gamma(\eta_2)}{2v_{13}}\right)^{-1}$.

Region 5c: This subregion falls in Regions d and e in Figure A.4. According to the function f in Regions d and e as well as Lemma 5, the system converges to a period-2 orbit: s_1 and $\gamma(s_1)$. When the system operates on the period-2 orbit, according to the proof of Lemma 4 (see Regions dand e), W_1 is halted at location s_1 for every other hand-off. The average throughput is $\mathcal{T} = 2\left(\frac{1-s_1-s_3}{v_{22}} + \frac{s_1-\gamma(s_1)}{v_{23}} + \frac{s_1-\gamma(s_1)}{v_{21}} + \frac{1-s_1-s_3}{v_{22}}\right)^{-1} = \left[\frac{s_1-\gamma(s_1)}{2} \cdot \left(\frac{1}{v_{21}} + \frac{1}{v_{23}}\right) + \frac{1-s_1-s_3}{v_{22}}\right]^{-1}$.

Region 5d: This subregion falls in Region b in Figure A.4. According to the function f in Region b as well as Lemma 5, the system converges to a period-2 orbit: η_1 and $s_1 - s_3$. When the system operates on the period-2 orbit, according to the proof of Lemma 4 (see Region b), W_1 is blocked at location 0 for every other hand-off, and W_2 is halted at location 1 for every other hand-off. The average throughput is $\mathcal{T} = 2\left(\frac{s_3}{v_{21}} + \frac{\eta_1}{v_{11}} + \frac{s_3-s_1+\eta_1}{v_{13}} + \frac{s_1-s_3}{v_{11}}\right)^{-1} = \left(\frac{s_1-s_3+\eta_1}{2v_{11}} + \frac{s_3-s_1+\eta_1}{2v_{13}} + \frac{s_3}{2v_{21}}\right)^{-1}$.

Region 5e: This subregion falls in Region c in Figure A.4. According to the function f in Regionc as well as Lemma 5, the system converges to a period-2 orbit: η_1 and η_2. When the system operates on the period-2 orbit, accordingtothe proof of Lemma 4(see Region c), W_1 is blocked at location 0 for every other hand-off, and W_2 is blocked at location $s_1 + s_2$ for every other hand-off. The average throughput is $\mathcal{T} = 2\left(\frac{s_3-s_1+\eta_1}{v_{13}} + \frac{\eta_2}{v_{11}} + \frac{s_1-\eta_2}{v_{21}} + \frac{\eta_1}{v_{11}}\right)^{-1} = \left(\frac{\eta_1+\eta_2}{2v_{11}} + \frac{s_3-s_1+\eta_1}{2v_{13}} + \frac{s_1-\eta_2}{2v_{21}}\right)^{-1}$.

Region 5f: This subregion falls in Region e in Figure A.4. According to the function f in Regione as well as Lemma 5, the system converges to a period-2 orbit: s_1 and η_2. When the system operates on the period-2 orbit, accordingtothe proof of

Lemma 4 (see Region e), W_1 is halted at location s_1 for every other hand-off, and W_2 is blocked at location $s_1 + s_2$ for every other hand-off. The average throughput is $\mathcal{T} = 2\left(\frac{s_3}{v_{13}} + \frac{\eta_2}{v_{11}} + \frac{s_1-\eta_2}{v_{21}} + \frac{1-s_1-s_3}{v_{22}}\right)^{-1} = \left(\frac{\eta_2}{2v_{11}} + \frac{s_3}{2v_{13}} + \frac{s_1-\eta_2}{2v_{21}} + \frac{1-s_1-s_3}{2v_{22}}\right)^{-1}$.

Region 5g: This subregion falls in Region d in Figure A.4. According to the function f in Region d as well as Lemma 5, the system converges to a period-2 orbit: s_1 and $s_1 - s_3$. When the system operates on the period-2 orbit, according to the proof of Lemma 4 (see Region d), W_1 is first blocked at location 0 and then halted at location s_1 for every other hand-off, and W_2 is halted at location 1 for every other hand-off. The average throughput is $\mathcal{T} = 2\left(\frac{s_3}{v_{13}} + \frac{s_1-s_3}{v_{11}} + \frac{s_3}{v_{21}} + \frac{1-s_1-s_3}{v_{22}}\right)^{-1} = \left(\frac{s_1-s_3}{2v_{11}} + \frac{s_3}{2v_{13}} + \frac{s_3}{2v_{21}} + \frac{1-s_1-s_3}{2v_{22}}\right)^{-1}$. □

A.3 Technical details for the M-station case in Chapter 4

In the M-station U-line, a hand-off position falls in the range $[\underline{x}, \bar{x}]$, where $\underline{x} = s_1 - s_3$ and $\bar{x} = s_1$. Recall that after a hand-off, W_1 first works on stage 3 before he works on stage 1 and W_2 first works on stage 1, and then works on stages 2 and 3 (see Figure 4.6). Note that W_1 can only be blocked at the start of a station in stage 1 and can only be halted at location s_1, and W_2 can only be blocked at the start of a station in stage 3 and can only be halted at location 1. Let $L_j(k)$ denote the location of the start of $S_j(k)$ on the conceptual line, for $k = 1, \ldots, m_j$ and $j = 1, 2, 3$.

Consider W_i starts from a hand-off position x. For convenience, we say W_i is *blocked at $L_j(k)$ from x* if he is blocked at $L_j(k)$ before the next hand-off. Similarly, we say W_i is *halted at location L from x* if he is halted at location L before the next hand-off.

We have the following properties:

Property 1. *For any x and x' in $[\underline{x}, \bar{x}]$, if W_i is blocked at $L_j(k)$ from both x and x', then $f(x) = f(x')$.*

Property 2. *For any x and x' in $[\underline{x}, \bar{x}]$, if W_i is halted at location L from both x and x', then $f(x) = f(x')$.*

Property3. *For any x and x' in $[\underline{x}, \bar{x}]$ such that $x > x'$, if W_1 is blocked at $L_1(k)$ from x then W_1 is blocked at $L_1(k)$ from x', and if W_1 is halted at location s_1 from x then W_1 is halted at location s_1 from x'.*

Property4. *For any x and x' in $[\underline{x}, \bar{x}]$ such that $x < x'$, if W_2 is blocked at $L_3(k)$ from x then W_2 is blocked at $L_3(k)$ from x', and if W_2 is halted at location 1 from x then W_2 is halted at location 1 from x'.*

A.3.1 Characterizing the function f

From the above properties, we have the following results.

Lemma 8. *There exists a constant c_1 such that W_1 is blocked or halted from any $x \in [\underline{x}, c_1)$, but he is neither blocked nor halted from any $x \in [c_1, \bar{x})$.*

Proof. The lemma claims that W_1 is blocked or halted from any $x < c_1$. We can find c_1 in each of the following three cases: (1) If W_1 is neither blocked nor halted from $\underline{x}$, then according to Property 3, W_1 is neither blocked nor halted from any $x \in [\underline{x}, \bar{x}]$. Thus, we have $c_1 = \underline{x}$. (2) If W_1 is blocked or halted from $\bar{x}$, then according to Property 3, W_1 is blocked or halted from any $x \in [\underline{x}, \bar{x}]$. Thus, we have $c_1 = \bar{x}$. (3) Otherwise, according to Property 3, there exists a hand-off position c_1 such that W_1 is blocked or halted from any $x \in [\underline{x}, c_1)$, but he is neither blocked nor halted from any $x \in [c_1, \bar{x}]$. The three cases above imply that W_1 is blocked or halted from any $x \in [\underline{x}, c_1)$, but he is neither blocked nor halted from any $x \in [c_1, \bar{x})$. $\square$

Using Property 4, we can prove the following lemma similarly.

Lemma 9. *There exists a constant c_2 such that W_2 is blocked or halted from any $x \in (c_2, \bar{x}]$, but he is neither blocked nor halted from any $x \in (\underline{x}, c_2]$.*

Proof. We then prove part (II), which claims that W_2 is blocked or halted from any $x > c_2$. We can find c_2 in each of the following three cases: (1) If W_2 is blocked or halted from $\underline{x}$, then let $c_2 = \underline{x}$. According to Property 4, W_2 is blocked or halted from any $x \in [\underline{x}, \bar{x}]$. (2) If W_2 is neither blocked nor halted from $\bar{x}$, then let $c_2 = \bar{x}$. According to Property 4, W_2 is neither blocked nor halted from any $x \in [\underline{x}, \bar{x}]$. (3) Otherwise, according to Property 4, there exists a hand-off position c_2' such that W_2 is blocked or halted from any $x \in (c_2', \bar{x}]$, but he is neither blocked nor halted from any $x \in [\underline{x}, c_2']$. Let $c_2 = c_2'$. Thus, W_2 is blocked or halted from any $x \in [c_2, \bar{x}]$, but he is neither blocked nor halted from any $x \in (\underline{x}, c_2]$. As a result in all three cases W_2 is blocked or halted from any $x \in (c_2, \bar{x}]$, but he is neither blocked nor halted from any $x \in (\underline{x}, c_2]$. $\square$

Together with Properties 1 and 2, Lemmas 8 and 9 imply the following result.

Corollary 2. *There exist constants Y_1 and Y_2 such that for any $x \in [\underline{x}, c_1)$, $f(x) = Y_1$, and for any $x \in (c_2, \bar{x}]$, $f(x) = Y_2$.*

Lemma 10. *If $c_1 < c_2$ then f is strictly decreasing in $[c_1, c_2]$.*

Proof. According to Lemmas 8 and 9, both workers are neither blocked nor halted from any $x \in [c_1, c_2]$. For any hand-off positions χ_1 and χ_2 such that $c_1 \leq \chi_1 < \chi_2 \leq c_2$, we will show that $f(\chi_1) > f(\chi_2)$. There are two cases: (1) $f(\chi_2) \geq 0$; (2) $f(\chi_2) < 0$.

For case (1), it is sufficient to prove that after a hand-off at χ_1, when W_1 works in stage 1 and arrives at position $f(\chi_2)$ he has not met W_2. For any hand-off position $x \in [\underline{x}, \bar{x}]$, let $t_1(x)$ denote the total time for W_1 to start from x, finish his item at the end of stage 3, work on a new item in stage 1, and reach position $f(\chi_2)$. Let $t_2(x)$ denote the total time for W_2 to start from $\max\{0, x\}$, work on his item in stages 1, 2, and 3, and reach position $f(\chi_2)$. Since $\chi_1 < \chi_2$, we have $t_1(\chi_1) < t_1(\chi_2)$ and $t_2(\chi_1) \geq t_2(\chi_2)$. In addition, we know that $t_1(\chi_2) = t_2(\chi_2)$. Thus, we have $t_1(\chi_1) < t_1(\chi_2) = t_2(\chi_2) \leq t_2(\chi_1)$, which imply $f(\chi_1) > f(\chi_2)$.

For case (2), it is sufficient to prove that after a hand-offat position χ_1, when W_1 arrives at location 1, W_2 has not reached position $f(\chi_2)$. For any hand-off position $x \in [\underline{x}, \bar{x}]$, let $t_1(x)$ be the total time for W_1 to start from x and finish his item at the end of stage 3 (reach location 1). Let $t_2(x)$ denote the total time for W_2 to start from $\max\{0, x\}$, work on his item in stages 1, 2, and 3, and reach position $f(\chi_2)$. Since $\chi_1 < \chi_2$, we have $t_1(\chi_1) < t_1(\chi_2)$ and $t_2(\chi_1) \geq t_2(\chi_2)$. In addition, we know that $t_1(\chi_2) = t_2(\chi_2)$. Thus, we have $t_1(\chi_1) < t_1(\chi_2) = t_2(\chi_2) \leq t_2(\chi_1)$, which imply $f(\chi_1) > f(\chi_2)$. □

Lemma 11. *f is continuous.*

Proof. According to the proofs of Lemmas 8 and 9, W_1 is almost blocked or halted from c_1 and W_2 is almost blocked or halted from c_2. Together with Corollary 2, we have $f(c_1) = Y_1$ and $f(c_2) = Y_2$. Thus, it is sufficient to prove that f is continuous in $[c_1, c_2]$.

For convenience, define $v_{i,\max} = \max_{j,k} v_{ij}(k)$ and $v_{i,\min} = \min_{j,k} v_{ij}(k)$ for $i = 1, 2$.

Consider any hand-off positions χ_1 and χ_2, where $c_1 \le \chi_1 < \chi_2 \le c_2$ such that $\chi_2 - \chi_1 < \delta$ for a small δ. There are two cases: (1) $f(\chi_2) \ge 0$; (2) $f(\chi_2) < 0$. For each case, we adopt the same definitions of $t_1(x)$ and $t_2(x)$ as those in the proof of Lemma 10.

For case (1), the proof of Lemma 10 shows that after a hand-off at position χ_1, when W_1 works in stage 1 and arrives at position $f(\chi_2)$, he has not met W_2. Meanwhile, W_2 reaches the position $h_2 \le f(\chi_2) + v_{2,\max} \cdot (t_2(\chi_1) - t_1(\chi_1))$. Thus, we have $f(\chi_1) < h_2$.

For case (2), the proof of Lemma 10 shows that after a hand-off at position χ_1, when W_1 arrives at location 1, W_2 has not reached position $f(\chi_2)$. Instead, W_2 reaches the position $h_2 \le f(\chi_2) + v_{2,\max} \cdot (t_2(\chi_1) - t_1(\chi_1))$. Thus, we have $f(\chi_1) \le h_2$.

Combining cases (1) and (2), we have $f(\chi_1) \le h_2 \le f(\chi_2) + v_{2,\max} \cdot (t_2(\chi_1) - t_1(\chi_1))$. Since $t_2(\chi_1) - t_1(\chi_1) = (t_2(\chi_1) - t_2(\chi_2)) + (t_1(\chi_2) - t_1(\chi_1)) \le \frac{\chi_2 - \chi_1}{v_{2,\min}} + \frac{\chi_2 - \chi_1}{v_{1,\min}} = (\chi_2 - \chi_1) \cdot \left(\frac{1}{v_{1,\min}} + \frac{1}{v_{2,\min}}\right) < \left(\frac{1}{v_{1,\min}} + \frac{1}{v_{2,\min}}\right) \cdot \delta$, we have $f(\chi_1) - f(\chi_2) < v_{2,\max} \cdot \left(\frac{1}{v_{1,\min}} + \frac{1}{v_{2,\min}}\right) \cdot \delta$. Thus, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any hand-off positions χ_1 and χ_2, if $\chi_2 - \chi_1 < \delta$ then $f(\chi_1) - f(\chi_2) < \varepsilon$. Therefore, $f(x)$ is continuous in $[c_1, c_2]$. $\square$

For convenience, we say a hand-off position x is an *interior* hand-off position if the locations corresponding to x on the U-line fall in the interior of some stations.

Lemma 12. *f is piecewise linear.*

Proof. According to Corollary 2, it is sufficient to prove that f is piecewise linear in $[c_1, c_2]$. Consider any hand-off position $\chi \in (c_1, c_2)$, such that both χ and $f(\chi)$ are interior hand-off positions. We will show that f is linear in the neighborhood of such χ.

For convenience, define u_1 as the velocity of W_1 at position χ when he works in stage 3. If $f(\chi) \ge 0$, then define v_1 as the velocity of W_1 at position $f(\chi)$ when he works in stage 1. If $\chi \geq 0$, then define v_2 as the velocity of W_2 at position χ when he works in stage 1. We also define u_2 as the velocity of W_2 at position $f(\chi)$ when he works in stage 3.

For any $x = \chi \pm \Delta x$, where Δx is a small positive number, we have four cases: (1) $\chi > 0$ and $f(\chi) > 0$; (2) $\chi > 0$ and $f(\chi) < 0$; (3) $\chi < 0$ and $f(\chi) > 0$; (4) $\chi < 0$ and $f(\chi) < 0$. For case (1), we have $f(x) = f(\chi) \mp \frac{v_1 u_2}{v_1+u_2} \cdot \left(\frac{1}{u_1} + \frac{1}{v_2}\right) \cdot \Delta x$. For case (2), we have $f(x) = f(\chi) \mp u_2 \cdot \left(\frac{1}{u_1} + \frac{1}{v_2}\right) \cdot \Delta x$. For case (3), we have $f(x) = f(\chi) \mp \frac{v_1 u_2}{v_1+u_2} \cdot \frac{1}{u_1} \cdot \Delta x$. For case (4), we have $f(x) = f(\chi) \mp \frac{u_2}{u_1} \cdot \Delta x$. Thus, f is linear in the neighborhood of χ, and so f is piecewise linear in $[c_1, c_2]$. $\square$

The following corollary summarizes the above results.

Corollary 3. *f is continuous, non-increasing, and has the following form*

$$f(x) \quad = \quad \begin{cases} Y_1, & \text{if } x \in [\underline{x}, c_1); \\ F(x), & \text{if } x \in [c_1, c_2]; \\ Y_2, & \text{otherwise}; \end{cases}$$

where F is strictly decreasing and piecewise linear.

A.3.2 Asymptotic behaviors

Corollary 3 implies the following lemma.

Lemma 13. *There exists a unique fixed point and there are no periodic orbits of period greater than 2 in the system.*

Proof. According to Brouwer's fixed point theorem, there exists a fixed point because f is continuous. Since f is also non-increasing, the fixed point is unique.

We then prove by contradiction that there are no periodic orbits of period greater than 2. Suppose there exists a periodic orbit of period $\pi > 2$: $x_1, x_2, \ldots, x_\pi$. For convenience, define $X = \{x_1, x_2, \ldots, x_\pi\}$. First note that for any $x_i \in X$, we have $x_i \neq x^*$. Without loss of generality, assume that $x_1 < x^*$. Since f is non-increasing, for any $x_i \in X$, if $x_i < x^*$, then $f(x_i) > x^*$, and if $x_i > x^*$, then $f(x_i) < x^*$. As a result, we have $f^{2n-1}(x_1) > x^*$ and $f^{2n}(x_1) < x^*$, for $n = 1, 2, \ldots$. Thus, π is even because $f^\pi(x_1) = x_1 < x^*$.

Since $f^2(\cdot)$ is non-decreasing, if $x_1 < x_3$ then we have $x_1 < x_3 = f^2(x_1) < x_5 = f^2(x^3) < \cdots < x_1 = f^2(x_{\pi-1})$, which is a contradiction; otherwise, we have $x_1 > x_3 = f^2(x_1) > x_5 = f^2(x^3) > \cdots > x_1 = f^2(x_{\pi-1})$, which is also a contradiction. Therefore, there does not exist a periodic orbit of period $\pi > 2$. $\square$

The following lemma provides a sufficient condition for the fixed point x^* to be a global attractor. This condition can be tested easily. Let $P_j(k)$ denote the horizontal position of $L_j(k)$, for $k = 1, \ldots, m_j$ and $j = 1, 2, 3$. For convenience, define $P_1(m_1 + 1) = s_1$ and $P_3(m_3 + 1) = \underline{x}$.

Lemma 14. *The system converges to a fixed point x^* if for any pair of interior hand-off positions x and $f(x)$, one of the following conditions is satisfied:*

1. $x > 0$, $f(x) > 0$, *where* $x \in (P_1(k_1), P_1(k_1+1)) \bigcap (P_3(k_2+1), P_3(k_2))$ *and* $f(x) \in (P_1(k_3), P_1(k_3+1)) \bigcap (P_3(k_4+1), P_3(k_4))$, *and*
$$\frac{1}{v_{11}(k_3)} - \frac{1}{v_{13}(k_2)} > \frac{1}{v_{21}(k_1)} - \frac{1}{v_{23}(k_4)}.$$
2. $x > 0$, $f(x) < 0$, *where* $x \in (P_1(k_1), P_1(k_1+1)) \bigcap (P_3(k_2+1), P_3(k_2))$ *and* $f(x) \in (P_3(k_4+1), P_3(k_4))$, *and*
$$-\frac{1}{v_{13}(k_2)} > \frac{1}{v_{21}(k_1)} - \frac{1}{v_{23}(k_4)}.$$

3. $x < 0$, $f(x) > 0$, *where* $x \in (P_3(k_2+1), P_3(k_2))$ *and* $f(x) \in (P_1(k_3), P_1(k_3+1)) \bigcap (P_3(k_4+1), P_3(k_4))$, *and*

$$\frac{1}{v_{11}(k_3)} - \frac{1}{v_{13}(k_2)} > -\frac{1}{v_{23}(k_4)}.$$

4. $x < 0$, $f(x) < 0$, *where* $x \in (P_3(k_2+1), P_3(k_2))$ *and* $f(x) \in (P_3(k_4+1), P_3(k_4))$, *and*

$$-\frac{1}{v_{13}(k_2)} > -\frac{1}{v_{23}(k_4)}.$$

Proof. According to the proof of Lemma 12, the four conditions in Lemma 14 ensure that the absolute value of the derivative of f, where f is differentiable, is smaller than 1. Let $\rho \in [0,1)$ denote the largest absolute value of the slope of f. For any $x \in [\underline{x}, \bar{x}]$, we have $|f(x) - f(x^*)| \leq \rho|x - x^*|$. Since $f(x^*) = x^*$, we have $|f^n(x) - x^*| \leq \rho^n|x - x^*|$, and thus $\lim_{n\to\infty} f^n(x) = x^*$. Therefore, the system converges to the fixed point. □

If $s_1 \geq s_3$, then both interior hand-off positions x and $f(x)$ are positive. Lemma 14 reduces to the following result.

Corollary 4. *If* $s_1 \geq s_3$, *then the system converges to a fixed point* x^* *if for any pair of interior hand-off positions* x *and* $f(x)$, *where* $x \in (P_1(k_1), P_1(k_1+1)) \bigcap (P_3(k_2+1), P_3(k_2))$ *and* $f(x) \in (P_1(k_3), P_1(k_3+1)) \bigcap (P_3(k_4+1), P_3(k_4))$, *the following condition holds:*

$$\frac{1}{v_{11}(k_3)} - \frac{1}{v_{13}(k_2)} > \frac{1}{v_{21}(k_1)} - \frac{1}{v_{23}(k_4)}.$$

The following lemma gives a necessary and sufficient condition for the fixed point to be a global attractor. It is expensive to check this condition as we do not know the exact form of the function f.

Lemma 15. *The system converges to a fixed point* x^* *if and only if* $\lim_{n\to\infty} f^n(c_i) = x^*$ *for* $i = 1, 2$.

Proof. If the system converges to x^*, then $\lim_{n\to\infty} f^n(c_i) = x^*$, $i = 1, 2$. Thus, we only need to show the reverse: If $\lim_{n\to\infty} f^n(c_i) = x^*$ for $i = 1, 2$, then for any $x \in [\underline{x}, \bar{x}]$, $\lim_{n\to\infty} f^n(x) = x^*$.

For any $x \in [\underline{x}, c_1)$ we have $f(x) = f(c_1)$, and thus $\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} f^n(c_1) = x^*$. Similarly, for any $x \in (c_2, \bar{x}]$ we have $\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} f^n(c_2) = x^*$. For $n' = 1, 2, 3, \ldots$, let $I_{2n'-1}$ denote the interval $[f^{2n'-1}(c_2), f^{2n'-1}(c_1)]$ and let $I_{2n'}$ denote the interval $[f^{2n'}(c_1), f^{2n'}(c_2)]$. Since f is non-increasing and continuous, for any $x \in [c_1, c_2]$ we have $f^n(x) \in I_n$. Thus, for any $x \in [c_1, c_2]$, $\lim_{n\to\infty} f^n(x) \in \lim_{n\to\infty} I_n = [x^*, x^*]$, and so $\lim_{n\to\infty} f^n(x) = x^*$. $\square$

In the following sections, we provide the algorithms to compute the points c_1 and c_2, the fixed point x^*, and the throughput on x^*.

A.3.3 Computing the fixedpoint andthroughput

Algorithmic computationof c_1 and c_2

Let $t_{ij}(k)$ be the time for W_i to finish the work on $S_j(k)$: $t_{ij}(k) = s_j^k / v_{ij}(k)$.

Computation of c_1:

According to Lemma 8, if W_1 is neither blocked nor halted from $\underline{x}$ then $c_1 = \underline{x}$. If W_1 is blocked or halted from $\bar{x}$ then $c_1 = \bar{x}$. Otherwise, we compute c_1 as follows.

Recall from Lemma 8 that W_1 is blocked or halted from all $x \in [\underline{x}, c_1)$ but is neither blocked nor halted from all $x \in [c_1, \bar{x}]$. We can find an index q_1 such that W_1 is blocked or halted from the end of $S_3(q_1)$ (that is, point $P_3(q_1+1)$), but is neither blocked nor halted from the start of $S_3(q_1)$ (that is, point $P_3(q_1)$). Thus, we have $P_3(q_1+1) < c_1 \leq P_3(q_1)$.

There are three possible cases: (1) If $s_1 < s_3$ and W_1 is neither blocked nor halted from 0 then we have $c_1 \leq 0$. (2) If $s_1 < s_3$ and W_1 is blocked or halted from 0 then we have $c_1 > 0$. (3) If $s_1 \geq s_3$ then we have $c_1 > \underline{x} \geq 0$. For cases (2) and (3), since $c_1 > 0$, we can find an index p_1 such that the horizontal position c_1 corresponds to a location in station $S_1(p_1)$ (that is, $P_1(p_1) < c_1 \leq P_1(p_1+1)$); but there is no such an index for case (1).

For case (1), we have two scenarios.

- If W_1 is halted at location s_1 from $P_3(q_1+1)$, then W_1 is almost halted at s_1 from c_1: After a hand-off at c_1, W_1 arrives at location s_1 at the same time as W_2 reaches location s_1+s_2. We have

$$\frac{c_1 - P_3(q_1+1)}{v_{13}(q_1)} + \sum_{k=q_1+1}^{m_3} t_{13}(k) + \sum_{k=1}^{m_1} t_{11}(k) = \sum_{k=1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k)$$
$$\Rightarrow \quad c_1 = P_3(q_1+1) + v_{13}(q_1) \cdot \left(\sum_{k=1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) - \sum_{k=q_1+1}^{m_3} t_{13}(k) - \sum_{k=1}^{m_1} t_{11}(k) \right).$$

- If W_1 is not halted at location s_1 from $P_3(q_1+1)$, then let $S_1(r_1)$ be the last station such that W_1 is blocked at $L_1(r_1)$ from $P_3(q_1+1)$. W_1 is almost blocked at $L_1(r_1)$ from c_1: After a hand-off at c_1, W_1 arrives at location $L_1(r_1)$ at the same time as W_2 reaches location $L_1(r_1+1)$. We have

$$\frac{c_1 - P_3(q_1+1)}{v_{13}(q_1)} + \sum_{k=q_1+1}^{m_3} t_{13}(k) + \sum_{k=1}^{r_1-1} t_{11}(k) = \sum_{k=1}^{r_1} t_{21}(k)$$
$$\Rightarrow \quad c_1 = P_3(q_1+1) + v_{13}(q_1) \cdot \left(\sum_{k=1}^{r_1} t_{21}(k) - \sum_{k=q_1+1}^{m_3} t_{13}(k) - \sum_{k=1}^{r_1-1} t_{11}(k) \right).$$

For cases (2) and (3), we also have two scenarios.

- If W_1 is halted at location s_1 from $P_3(q_1+1)$, then W_1 is almost halted at s_1 from c_1: After a hand-off at c_1, W_1 arrives at location s_1 at the same time as W_2 reaches location s_1+s_2. We have

$$\begin{aligned}
&\frac{c_1 - P_3(q_1+1)}{v_{13}(q_1)} + \sum_{k=q_1+1}^{m_3} t_{13}(k) + \sum_{k=1}^{m_1} t_{11}(k) \\
&= \frac{P_1(p_1+1) - c_1}{v_{21}(p_1)} + \sum_{k=p_1+1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) \\
\Rightarrow \quad c_1 = &\left(\frac{1}{v_{13}(q_1)} + \frac{1}{v_{21}(p_1)}\right)^{-1} \cdot \left(\frac{P_3(q_1+1)}{v_{13}(q_1)} + \frac{P_1(p_1+1)}{v_{21}(p_1)} + \sum_{k=p_1+1}^{m_1} t_{21}(k)\right. \\
&\left. + \sum_{k=1}^{m_2} t_{22}(k) - \sum_{k=q_1+1}^{m_3} t_{13}(k) - \sum_{k=1}^{m_1} t_{11}(k)\right).
\end{aligned}$$

- If W_1 is not halted at location s_1 from $P_3(q_1+1)$, then let $S_1(r_1)$ be the last station such that W_1 is blocked at $L_1(r_1)$ from $P_3(q_1+1)$. W_1 is almost blocked at $L_1(r_1)$ from c_1: After a hand-off at c_1, W_1 arrives at location $L_1(r_1)$ at the same time as W_2 reaches location $L_1(r_1+1)$. We have

$$\begin{aligned}
&\frac{c_1 - P_3(q_1+1)}{v_{13}(q_1)} + \sum_{k=q_1+1}^{m_3} t_{13}(k) + \sum_{k=1}^{r_1-1} t_{11}(k) = \frac{P_1(p_1+1) - c_1}{v_{21}(p_1)} + \sum_{k=p_1+1}^{r_1} t_{21}(k) \\
\Rightarrow \quad c_1 = &\left(\frac{1}{v_{13}(q_1)} + \frac{1}{v_{21}(p_1)}\right)^{-1} \cdot \left(\frac{P_3(q_1+1)}{v_{13}(q_1)} + \frac{P_1(p_1+1)}{v_{21}(p_1)} + \sum_{k=p_1+1}^{r_1} t_{21}(k)\right. \\
&\left. - \sum_{k=q_1+1}^{m_3} t_{13}(k) - \sum_{k=1}^{r_1-1} t_{11}(k)\right).
\end{aligned}$$

Computation of c_2:

We compute c_2 in a similar way. According to Lemma 9, if W_2 is blocked or halted from $\underline{x}$ then $c_2 = \underline{x}$. If W_2 is neither blocked nor halted from $\bar{x}$ then $c_2 = \bar{x}$. Otherwise, we compute c_2 as follows.

Recall from Lemma 9 that W_2 is blocked or halted from all $x \in (c_2, \bar{x}]$ but is neither blocked nor halted from all $x \in [\underline{x}, c_2]$. Thus, we can find an index q_2 such that W_2 is blocked or halted from the start of $S_3(q_2)$ (that is, point $P_3(q_2)$), but is neither blocked nor halted from the end of $S_3(q_2)$ (that is, point $P_3(q_2+1)$). Thus, we have $P_3(q_2+1) \leq c_2 < P_3(q_2)$.

There are three possible cases: (1) If $s_1 < s_3$ and W_2 is blocked from $P_1(1)$ then we have $c_2 < 0$. (2) If $s_1 < s_3$ and W_2 is not blocked from $P_1(1)$ then we have $c_2 \geq 0$. (3) If $s_1 \geq s_3$ then we have $c_2 > \underline{x} \geq 0$. For cases (2) and (3), since $c_2 \geq 0$, we can find an index p_2 such that the horizontal position c_2 corresponds to a location in station $S_1(p_2)$ (that is, $P_1(p_2) \leq c_2 < P_1(p_2+1)$); but there is no such an index for case (1).

For case (1), W_2 is never halted. Let $S_3(r_2)$ be the last station such that W_2 is blocked at $L_3(r_2)$ from $P_3(q_2)$. W_2 is almost blocked at $L_3(r_2)$ from c_2: After a hand-off at c_2, W_2 arrives at location $L_3(r_2)$ at the same time as W_2 reaches location $L_3(r_2+1)$. We have

$$\frac{c_2 - P_3(q_2+1)}{v_{13}(q_2)} + \sum_{k=q_2+1}^{r_2} t_{13}(k) = \sum_{k=1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{r_2-1} t_{23}(k)$$
$$\Rightarrow \quad c_2 = P_3(q_2+1) + v_{13}(q_2) \cdot \left(\sum_{k=1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{r_2-1} t_{23}(k) - \sum_{k=q_2+1}^{r_2} t_{13}(k) \right).$$

For cases (2) and (3), we have two scenarios.

- If W_2 is halted at location 1 from $P_3(q_2)$, then W_2 is almost halted at location c_2: After a hand-off at c_2, W_2 arrives at location 1 at the same time as W_1 reaches location $s_1 - s_3$. Suppose $\underline{x} = s_1 - s_3 \in [P_1(r_0), P_1(r_0+1))$. We have

$$\begin{aligned}
&\frac{c_2 - P_3(q_2+1)}{v_{13}(q_2)} + \sum_{k=q_2+1}^{m_3} t_{13}(k) + \sum_{k=1}^{r_0-1} t_{11}(k) + \frac{\underline{x} - P_1(r_0)}{v_{11}(r_0)} \\
&\quad = \frac{P_1(p_2+1) - c_2}{v_{21}(p_2)} + \sum_{k=p_2+1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{m_3} t_{23}(k) \\
\Rightarrow \quad c_2 &= \left(\frac{1}{v_{13}(q_2)} + \frac{1}{v_{21}(p_2)}\right)^{-1} \cdot \left(\frac{P_3(q_2+1)}{v_{13}(q_2)} + \frac{P_1(p_2+1)}{v_{21}(p_2)}\right. \\
&\quad + \sum_{k=p_2+1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{m_3} t_{23}(k) \\
&\quad \left. - \sum_{k=q_2+1}^{m_3} t_{13}(k) - \sum_{k=1}^{r_0-1} t_{11}(k) - \frac{\underline{x} - P_1(r_0)}{v_{11}(r_0)}\right).
\end{aligned}$$

- If W_2 is not halted at location 1 from $P_3(q_2)$, then let $S_3(r_2)$ be the last station such that W_2 is blocked at $L_3(r_2)$ from $P_3(q_2)$. W_2 is almost blocked at $L_3(r_2)$ from c_2: After a hand-off at c_2, W_2 arrives at location $L_3(r_2)$ at the same time as W_1 reaches location $L_3(r_2+1)$. We have

$$\begin{aligned}
&\frac{c_2 - P_3(q_2+1)}{v_{13}(q_2)} + \sum_{k=q_2+1}^{r_2} t_{13}(k) \\
&\quad = \frac{P_1(p_2+1) - c_2}{v_{21}(p_2)} + \sum_{k=p_2+1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{r_2-1} t_{23}(k) \\
\Rightarrow \quad c_2 &= \left(\frac{1}{v_{13}(q_2)} + \frac{1}{v_{21}(p_2)}\right)^{-1} \cdot \left(\frac{P_3(q_2+1)}{v_{13}(q_2)} + \frac{P_1(p_2+1)}{v_{21}(p_2)}\right. \\
&\quad \left. + \sum_{k=p_2+1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{r_2-1} t_{23}(k) - \sum_{k=q_2+1}^{r_2} t_{13}(k)\right).
\end{aligned}$$

Algorithmic computationof x^* andthe average throughput

We compute x^* for each of the following four cases:

Case (A): $c_1 = \bar{x}$

W_1 is constantly halted at location s_1 from any $x < c_1$, and thus $x^* = \bar{x} = s_1$.

Since W_2 is neither blocked nor halted from $x^* = \bar{x}$, the throughput $\mathcal{T} = \left(\sum_{k=1}^{m_2} t_{22}(k)\right)^{-1}$.

Case (B): $c_2 < c_1 < \bar{x}$

Corollary 3 implies that f is a constant. For any $x \in (c_2, c_1)$, according to Lemmas 8 and 9, both workers are blocked or halted from x. Note that if W_2 is blocked from x then W_1 is neither blocked nor halted from x. Since W_1 is blocked or halted from x, we know that W_2 is not blocked from x. Thus, W_2 is halted at location 1 from x, and so $f(x) = \underline{x} = s_1 - s_3$ for any $x \in (c_2, c_1)$. As a result, the fixed point is $x^* = \underline{x}$.

Let p^* be the index such that $\underline{x} \in (P_1(p^*), P_1(p^*+1)]$. W_1 is not halted at location s_1 from $\underline{x}$ but may be blocked at $L_1(p^*)$ from $\underline{x}$. If W_1 is blocked at $L_1(p^*)$, that is, if $\sum_{k=1}^{p^*-1} t_{11}(k) < \frac{P_1(p^*+1)-\underline{x}}{v_{21}(p^*)}$, then the throughput $\mathcal{T} = \left(\frac{P_1(p^*+1)-\underline{x}}{v_{21}(p^*)} + \frac{\underline{x}-P_1(p^*)}{v_{11}(p^*)}\right)^{-1}$. Otherwise, W_1 is neither blocked nor halted, and thus the throughput $\mathcal{T} = \left(\sum_{k=1}^{p^*-1} t_{11}(k) + \frac{\underline{x}-P_1(p^*)}{v_{11}(p^*)}\right)^{-1}$.

Case (C): $c_2 = c_1 < \bar{x}$

Corollary 3 implies that f is a constant. W_1 is blocked or halted from any $x < c_1$, and W_2 is blocked or halted from any $x > c_2 = c_1$. Similar to the analysis for case (B), since both workers are almost blocked or halted from c_1, W_2 is almost halted at location 1 from c_1. Thus, $f(c_1) = \underline{x}$. Since f is a constant, the fixed point is $x^* = \underline{x}$.

We can compute the throughput in the same way as in case (B): If $\sum_{k=1}^{p^*-1} t_{11}(k) < \frac{P_1(p^*+1)-\underline{x}}{v_{21}(p^*)}$, then the throughput $\mathcal{T} = \left(\frac{P_1(p^*+1)-\underline{x}}{v_{21}(p^*)} + \frac{\underline{x}-P_1(p^*)}{v_{11}(p^*)}\right)^{-1}$; otherwise, $\mathcal{T} = \left(\sum_{k=1}^{p^*-1} t_{11}(k) + \frac{\underline{x}-P_1(p^*)}{v_{11}(p^*)}\right)^{-1}$.

Case (D): $c_1 < c_2 \leq \bar{x}$

There are three subcases (1) $x^* < c_1$, (2) $c_1 \leq x^* \leq c_2$, and (3) $x^* > c_2$, which are equivalent to (1) $Y_1 < c_1$, (2) $Y_1 \geq c_1$ and $Y_2 \leq c_2$, and (3) $Y_2 > c_2$ respectively. In the rest of the algorithm, we first compute x^* for cases (1) and (3), and then for case (2).

The fixed points are Y_1 and Y_2 for cases (1) and (3) respectively. To compute Y_i, $i = 1, 2$, we can find the next hand-off position after a hand-off at c_i because $f(c_i) = Y_i$. Lemmas 8 and 9 imply that W_i is neither blocked nor halted from c_i. Recall from the computation of c_1 and c_2 (Section A.3.3) that c_i falls in the interval $[P_3(q_i+1), P_3(q_i)]$. Furthermore, if $c_i > 0$ then c_i also falls in the interval $[P_1(p_i), P_1(p_i+1)]$. Suppose Y_i is in the interval $[P_3(q_i^*+1), P_3(q_i^*)]$. The two workers spend an equal amount of time between a hand-off at c_i and the next hand-off at Y_i, for $i = 1, 2$. There are four cases:

(a) $c_i \leq 0$ and $Y_i \leq 0$: We have

$$\begin{aligned}\frac{c_i - P_3(q_i+1)}{v_{13}(q_i)} + \sum_{k=q_i+1}^{m_3} t_{13}(k) &= \sum_{k=1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{q_i^*-1} t_{23}(k) \\ &\quad + \frac{P_3(q_i^*) - Y_i}{v_{23}(q_i^*)} \\ \Rightarrow \quad Y_i = P_3(q_i^*) + v_{23}(q_i^*) \cdot &\left(\sum_{k=1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{q_i^*-1} t_{23}(k) \right. \\ &\quad \left. - \frac{c_i - P_3(q_i+1)}{v_{13}(q_i)} - \sum_{k=q_i+1}^{m_3} t_{13}(k) \right).\end{aligned}$$

(b) $c_i > 0$ and $Y_i \leq 0$: We have

$$\begin{aligned}&\frac{c_i - P_3(q_i+1)}{v_{13}(q_i)} + \sum_{k=q_i+1}^{m_3} t_{13}(k) \\ &= \frac{P_1(p_i+1) - c_i}{v_{21}(p_i)} + \sum_{k=p_i+1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{q_i^*-1} t_{23}(k) + \frac{P_3(q_i^*) - Y_i}{v_{23}(q_i^*)} \\ \Rightarrow \quad &Y_i = P_3(q_i^*) + v_{23}(q_i^*) \cdot \left(\frac{P_1(p_i+1) - c_i}{v_{21}(p_i)} + \sum_{k=p_i+1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) \right. \\ &\quad \left. + \sum_{k=1}^{q_i^*-1} t_{23}(k) - \frac{c_i - P_3(q_i+1)}{v_{13}(q_i)} - \sum_{k=q_i+1}^{m_3} t_{13}(k) \right).\end{aligned}$$

(c) $c_i \le 0$ and $Y_i > 0$: Suppose Y_i falls in $[P_1(p_i^*), P_1(p_i^*+1)]$. We have

$$\begin{aligned}
&\frac{c_i - P_3(q_i+1)}{v_{13}(q_i)} + \sum_{k=q_i+1}^{m_3} t_{13}(k) + \sum_{k=1}^{p_i^*-1} t_{11}(k) + \frac{Y_i - P_1(p_i^*)}{v_{11}(p_i^*)} \\
&= \sum_{k=1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{q_i^*-1} t_{23}(k) + \frac{P_3(q_i^*) - Y_i}{v_{23}(q_i^*)} \\
\Rightarrow \quad Y_i &= \left(\frac{1}{v_{11}(p_i^*)} + \frac{1}{v_{23}(q_i^*)}\right)^{-1} \cdot \left(\frac{P_1(p_i^*)}{v_{11}(p_i^*)} + \frac{P_3(q_i^*)}{v_{23}(q_i^*)}\right. \\
&\quad + \sum_{k=1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{q_i^*-1} t_{23}(k) \\
&\quad \left. - \frac{c_i - P_3(q_i+1)}{v_{13}(q_i)} - \sum_{k=q_i+1}^{m_3} t_{13}(k) - \sum_{k=1}^{p_i^*-1} t_{11}(k)\right).
\end{aligned}$$

(d) $c_i > 0$ and $Y_i > 0$: Suppose Y_i falls in $[P_1(p_i^*), P_1(p_i^*+1)]$. We have

$$\begin{aligned}
&\frac{c_i - P_3(q_i+1)}{v_{13}(q_i)} + \sum_{k=q_i+1}^{m_3} t_{13}(k) + \sum_{k=1}^{p_i^*-1} t_{11}(k) + \frac{Y_i - P_1(p_i^*)}{v_{11}(p_i^*)} \\
&= \frac{P_1(p_i+1) - c_i}{v_{21}(p_i)} + \sum_{k=p_i+1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{q_i^*-1} t_{23}(k) + \frac{P_3(q_i^*) - Y_i}{v_{23}(q_i^*)} \\
\Rightarrow \quad Y_i &= \left(\frac{1}{v_{11}(p_i^*)} + \frac{1}{v_{23}(q_i^*)}\right)^{-1} \cdot \left(\frac{P_1(p_i^*)}{v_{11}(p_i^*)} + \frac{P_3(q_i^*)}{v_{23}(q_i^*)}\right. \\
&\quad + \frac{P_1(p_i+1) - c_i}{v_{21}(p_i)} + \sum_{k=p_i+1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{q_i^*-1} t_{23}(k) \\
&\quad \left. - \frac{c_i - P_3(q_i+1)}{v_{13}(q_i)} - \sum_{k=q_i+1}^{m_3} t_{13}(k) - \sum_{k=1}^{p_i^*-1} t_{11}(k)\right).
\end{aligned}$$

For case (2), we have $x^* \in [c_1, c_2]$, and so each worker is neither blocked nor halted from x^* due to Lemmas 8 and 9. Suppose x^* is in the interval $[P_3(q_3^*+1), P_3(q_3^*)]$. The two workers spend an equal amount of time between two successive hand-offs at x^*. If

$x^* \leq 0$, then we have

$$\frac{x^* - P_3(q_3^*+1)}{v_{13}(q_3^*)} + \sum_{k=q_3^*+1}^{m_3} t_{13}(k) = \sum_{k=1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{q_3^*-1} t_{23}(k) + \frac{P_3(q_3^*) - x^*}{v_{23}(q_3^*)}$$

$$\Rightarrow \quad x^* = \left(\frac{1}{v_{13}(q_3^*)} + \frac{1}{v_{23}(q_3^*)}\right)^{-1} \cdot \Bigg(\frac{P_3(q_3^*+1)}{v_{13}(q_3^*)} + \frac{P_3(q_3^*)}{v_{23}(q_3^*)} + \sum_{k=1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{q_3^*-1} t_{23}(k) - \sum_{k=q_3^*+1}^{m_3} t_{13}(k)\Bigg).$$

Otherwise, we have $x^* > 0$. Suppose $x^* \in [P_1(p_3^*), P_1(p_3^*+1)]$. We have

$$\frac{x^* - P_3(q_3^*+1)}{v_{13}(q_3^*)} + \sum_{k=q_3^*+1}^{m_3} t_{13}(k) + \sum_{k=1}^{p_3^*-1} t_{11}(k) + \frac{x^* - P_1(p_3^*)}{v_{11}(p_3^*)}$$

$$= \frac{P_1(p_3^*+1) - x^*}{v_{21}(p_3^*)} + \sum_{k=p_3^*+1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{q_3^*-1} t_{23}(k) + \frac{P_3(q_3^*) - x^*}{v_{23}(q_3^*)}$$

$$\Rightarrow \quad x^* = \left(\frac{1}{v_{11}(p_3^*)} + \frac{1}{v_{13}(q_3^*)} + \frac{1}{v_{21}(p_3^*)} + \frac{1}{v_{23}(q_3^*)}\right)^{-1} \cdot \Bigg(\frac{P_1(p_3^*)}{v_{11}(p_3^*)} + \frac{P_3(q_3^*+1)}{v_{13}(q_3^*)} + \frac{P_1(p_3^*+1)}{v_{21}(p_3^*)} + \frac{P_3(q_3^*)}{v_{23}(q_3^*)} + \sum_{k=p_3^*+1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{q_3^*-1} t_{23}(k) - \sum_{k=q_3^*+1}^{m_3} t_{13}(k) - \sum_{k=1}^{p_3^*-1} t_{11}(k)\Bigg).$$

Now we start to compute the throughput. Since in case (D) we have $c_1 < c_2 \leq \bar{x}$, for any $x^* \in [\underline{x}, \bar{x}]$, x^* falls in $[c_1, \bar{x}]$ or $[\underline{x}, c_2]$. According to the proofs of Lemmas 8 and 9, at least one of the workers is neither blocked nor halted from x^*:

(i) W_1 is neither blocked nor halted from x^*. If $x^* \leq 0$ then the throughput $\mathcal{T} = \left(\frac{x^* - P_3(q_3^*+1)}{v_{13}(q_3^*)} + \sum_{k=q_3^*+1}^{m_3} t_{13}(k)\right)^{-1}$. Otherwise, the throughput $\mathcal{T} = \left(\frac{x^* - P_3(q_3^*+1)}{v_{13}(q_3^*)} + \sum_{k=q_3^*+1}^{m_3} t_{13}(k) + \sum_{k=1}^{p_3^*-1} t_{11}(k) + \frac{x^* - P_1(p_3^*)}{v_{11}(p_3^*)}\right)^{-1}$.

(ii) W_2 is neither blocked nor halted from x^*. If $x^* \leq 0$ then the throughput $\mathcal{T} = \left(\sum_{k=1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{q_3^*-1} t_{23}(k) + \frac{P_3(q_3^*) - x^*}{v_{23}(q_3^*)}\right)^{-1}$. Otherwise, we have

$x^* > 0$. The throughput $\mathcal{T} = \left(\frac{P_1(p_3^*+1)-x^*}{v_{21}(p_3^*)} + \sum_{k=p_3^*+1}^{m_1} t_{21}(k) + \sum_{k=1}^{m_2} t_{22}(k) + \sum_{k=1}^{q_3^*-1} t_{23}(k) + \frac{P_3(q_3^*)-x^*}{v_{23}(q_3^*)}\right)^{-1}$.