SIMULATION-BASED ESTIMATION METHODS FOR CROSS-SECTIONAL FINANCIAL ASSET PRICING

LU CHENYAN

SINGAPORE MANAGEMENT UNIVERSITY

2010

Simulation-Based Estimation Methods for Cross-Sectional Financial Asset Pricing

by

Lu Chenyan

Submitted to the School of Economics in partial fulfillment of the requirements for the Degree of Master of Science in Economics

Supervisor: Prof. Jun Yu

Singapore Management University

2010

Abstract

This paper extends the simulation-based estimation method proposed by Phillips and Yu (2009) to the cross-sectional case. We examine their finite-sample performance by conducting Monte-Carlo simulations of this simulation-based method to both the time-series model and the cross-sectional model. The simulation results show that the proposed simulation-based estimator can always reduce the percentage bias over the respective MLE and OLS estimator. Meanwhile, they do not significantly increase the variance or RMSE over their correspondent MLE and OLS estimator.

Key Words: Simulation-based estimation method; Black-Scholes Model; Cross-sectional case

Contents

1 Introduction

There has been an explosion of theoretical work in financial economics over the last half century. How to price financial assets has been an important topic in financial economics. Prices of the financial derivatives depend on the prices of the underlying assets, whose prices are often assumed to follow a parametric model. Theoretically, the pricing formula of the derivative is a function of these parameters.

Since the parameters of the underlying asset are usually unknown, they are generally replaced by respective estimates in the derivative pricing formulas. As a result, the statistical properties of the theoretical derivative price estimates hinge on those of the parameter estimates. Hence, the choice of method for parameter estimation is important and the topic has received a great deal of attention in the literature (see, for example, Aït-Sahalia, 1999).

In the time-series case, when the model is correctly specified, the preferred estimation method for the parameters is Maximum Likelihood (ML). It is well known that, under mild regularity conditions, the Maximum Likelihood Estimator (MLE) has many desirable asymptotic properties: consistency, normality and efficiency. Moreover, due to the invariance principle, a function of MLE itself is a MLE and hence inherits all the nice asymptotic properties (Zehna, 1966). Despite its generally nice asymptotic properties, ML is not necessarily the best estimation method for financial asset prices in finite sample. First of all, the closed-form likelihood function is usually difficult to calculate. Second, even if we have the analytic likelihood function, since many financial time-series variables are highly persistent, the MLE may have poor finite-sample statistical properties. For example, it may have substantial bias. Third, because the derivative price is a nonlinear transformation of the system parameters, insertion of even unbiased estimators of these parameters into the pricing formulas will not assure unbiased estimation of a derivative price (Ingersoll, 1976). Phillips and Yu (2009) reported evidence of bias in the MLE of volatility models, especially in the worst scenarios where there is persistence and nonlinearity, such as a deep-out-of-the-money option price.

In the literature, a great deal of effort has been done to improve the finite-sample performance of the MLE. For example, Butler and Schachter (1986) proposed an estimator of Black-Scholes option price based on Taylor series expansion, which is shown by Knight and Satchell (1997) only unbiased for the at-the-money options. Phillips and Yu (2005) proposed a jackknife procedure to reduce the large finite-sample bias in the mean reversion parameter. Usually, we obtain the analytic bias function of MLE at the first step. In the second step, we remove it from the biased estimator with the hope that the variance of the bias-corrected estimator does not increase or slightly increase, so that the mean squared error (MSE) reduces. These two methods share a common property: they trade off the gain that maybe achieved in bias reduction with a loss that increases the variance.

The idea of simulation-based estimation method origins from the observation that if the traditional estimator of a derivative price is biased with the real data, then it will also be biased with simulated data. Simulations therefore enable the bias function to be calibrated for the specific model and sample size being used. From this calibrated function, a bias reduction procedure is constructed and leads to a simulation-based estimate. As a good substitution of the bias-corrected estimator, it has been widely and successfully used in estimating parameters of various financial time-series model. For example, they are used in the context of continuous-time model to address the problem of discretization bias, e.g., Duffie and Singleton (1993). This method is also useful for improving the finite sample performance of the traditional methods, e.g., MacKinnon and Smith (1998), and of the dynamic panel model, e.g., Gouriéroux, Phillips and Yu (2010).

The present paper follows the method proposed by Phillips and Yu (2009). In that paper, they introduce a new simulation-based methodology of estimating derivative prices that can achieve the bias reduction as well as variance reduction. However, Phillips and Yu (2009) only consider the time-series case, where the gold standard method of estimation is ML. In this paper, we extend the method to the cross-sectional case.

In the next section, we review the existing method and introduce the direct simulation-based method of both time-series case and cross-sectional case. Then in section 3, using simulated data, we show how to implement the proposed method in relation of MLE estimation of call option price in the context of Black-Scholes model and in relation of OLS estimation of call option price in the context of cross-sectional setting. Finally, section 4 concludes and discusses the direction of further research.

2 Estimation Methods for Derivative Prices

2.1 Maximum Likelihood Estimation (MLE)

Let $S(t)$ denote the price of the underlying asset. Assume $S(t)$ follows the stochastic differential equation:

$$dS(t) = \mu\big(S(t), t; \theta\big)dt + \sigma\big(S(t), t; \theta\big)dB(t) \tag{2.1}$$

where $B(t)$ is the standard Brownian motion, $\mu\big(S(t), t; \theta\big)$ is a given drift function, $\sigma\big(S(t), t; \theta\big)$ is some specified diffusion function, and θ is an unknown parameter or a vector of unknown parameters.

Our observation is a sequence of time-series data of stock price $S = (S_h, S_{2h}, ..., S_{nh})$ available over a time period $[0, T(= nh)]$, where h is the sampling interval. Usually, we simplify them as $S = (S_1, S_2, ..., S_n)$. And our goal is to price a financial asset whose payoff is contingent on the value of $S(t)$. Denote the price of the derivative price as $P(\theta)$.

A common strategy to estimate $P(\theta)$ is as follows: in the first step, based on the observed data $S = (S_1, S_2, ..., S_n)$, estimate the parameter vector θ from equation (1), and denote the estimates as $\hat{\theta}$. In the second step, plug $\hat{\theta}$ into the pricing formula, we get $\hat{P} = P(\hat{\theta})$.

Since the equation (1) has the Markov property, we can get the log-likelihood function as:

$$l(\theta) = \sum_{t=2}^{n} \ln f(S_t|S_{t-1}; \theta) \tag{2.2}$$

where $f(S_t|S_{t-1}; \theta)$ is the conditional density function of S_t given S_{t-1}. Maximizing the log-likelihood function (equation (2.2)) with respective to θ, we get $\hat{\theta}_n^{ML}$, which is consistent, asymptotically normal, and asymptotically efficient under mild regularity conditions for stationary dynamic models. Plug $\hat{\theta}_n^{ML}$ into $P(\theta)$, we get $P(\hat{\theta}_n^{ML})$. Because of the principle of invariance, i.e., a function of MLE itself is a MLE, denote $\hat{P}_n^{ML} = P(\hat{\theta}_n^{ML})$. Hence, $\hat{P}_n^{ML}$ automatically inherits all the desirable asymptotic properties of ML estimator: consistent, asymptotically efficient, and asymptotically normal.

The limit distribution of $\hat{\theta}_n^{ML}$ is given by

$$\sqrt{n}(\hat{\theta}_n^{ML} - \theta) \quad \xrightarrow{d} \quad N\big(0, I^{-1}(\theta)\big)$$

where $I(\theta)$ is the information matrix, and the MLE is considered optimal since it achieves the Cramér-Rao lower bound and has the highest estimation precision in the limit when $n \to \infty$. By the delta method, we get the asymptotic distribution of $\hat{P}_n^{ML}$,

$$\sqrt{n}\big(\hat{P}_n^{ML} - P(\theta)\big) \quad \xrightarrow{d} \quad N\big(0, V_P\big)$$

where

$$V_P = \frac{\partial P}{\partial \theta'} I^{-1}(\theta) \frac{\partial P}{\partial \theta}$$

Although the exact MLE of the option price has these nice asymptotic properties, it is not always the best. For example, usually the closed-form expression for the conditional density function $f(S_t|S_{t-1};\theta)$ is not available. Secondly, even if sometimes we obtain the closed-form likelihood function, the MLE of parameters $\hat{\theta}_n^{ML}$ may suffer substantial finite-sample bias due to the high persistency of most of financial time-series data S_t. Moreover, even if the MLE of parameters $\hat{\theta}_n^{ML}$ has little bias, insertion of the parameter into the pricing formula will not assure the unbiased estimation of the derivative price due to the high nonlinearity of the pricing formula.

Assume $P(\theta)$ to be twice differentiable and θ is a scalar. Using Taylor expansion to $\hat{P}_n^{ML} = P(\hat{\theta}_n^{ML})$ around the true value θ, we get

$$\hat{P}_n^{ML} = P(\hat{\theta}_n^{ML}) \approx P(\theta) + \frac{\partial P(\theta)}{\partial \theta}(\hat{\theta}_n^{ML} - \theta) + \frac{1}{2}\frac{\partial^2 P(\theta)}{\partial \theta^2}(\hat{\theta}_n^{ML} - \theta)^2$$

Taking expectation on both sides, we get

$$\begin{aligned} E(\hat{P}_n^{ML}) &\approx P(\theta) + \frac{\partial P(\theta)}{\partial \theta} E(\hat{\theta}_n^{ML} - \theta) + \frac{1}{2}\frac{\partial^2 P(\theta)}{\partial \theta^2} E(\hat{\theta}_n^{ML} - \theta)^2 && (2.3) \\ &= P(\theta) + \frac{\partial P(\theta)}{\partial \theta} E(\hat{\theta}_n^{ML} - \theta) + \frac{1}{2}\frac{\partial^2 P(\theta)}{\partial \theta^2} \text{MSE}(\hat{\theta}_n^{ML}) && (2.4) \end{aligned}$$

From Equation (2.4), we find 3 situations where $\hat{P}_n^{ML}$ has substantial bias: first of all, when $\hat{\theta}_n^{ML}$ is biased, i.e., $E(\hat{\theta}_n^{ML} - \theta) \neq 0$; second, when MSE$(\hat{\theta}_n^{ML})$ is large, which is unfortunately the typical case of small sample; third, when $P(\theta)$ is highly nonlinear and $\frac{\partial^2 P(\theta)}{\partial \theta^2}$ is large.

2.2 Direct Simulation-Based Method

2.2.1 Time-series Case

Smith (1993) proposed a simulation-based method named indirect inference to estimate the models where the analytical likelihood function is difficult to get.

The detailed steps of applying indirect inference to our model is as follows:

(1) Get $\hat{\theta}_n^{ML}$ which is the MLE of parameter obtained from the real data.

(2) For any given θ, from equation (1), we can get the simulated data of stock price, say, $\tilde{S}^k(\theta) = \{\tilde{S}_1^k, \tilde{S}_2^k, ..., \tilde{S}_n^k\}$, where $k = 1, 2, ..., K$ is the simulation path. In order to calibrate the finite-sample bias, we choose the number of observations in $\tilde{S}^k(\theta)$ to be the same as the number of the real data.

(3) According to the simulated data of the k-th simulated path, $\tilde{S}^k(\theta) = \{\tilde{S}_1^k, \tilde{S}_2^k, ..., \tilde{S}_n^k\}$, we get the MLE of θ, named $\tilde{\phi}_n^{ML,k}(\theta)$.

(4) Choose θ, so that the average behavior of $\tilde{\phi}_n^{ML,k}(\theta)$ is matched with $\hat{\theta}_n^{ML}$ obtained from the observed data, i.e.,

$$\hat{\theta}_{n,K}^{II} = \arg\min_{\theta\in\Theta} \left\| \hat{\theta}_n^{ML} - \frac{1}{K}\sum_{k=1}^{K} \tilde{\phi}_n^{ML,k}(\theta) \right\|$$

Intuitively, any bias occurs in $\hat{\theta}_n^{ML}$ will also be present in $\tilde{\phi}_n^{ML,k}(\theta)$. So, when K goes to infinity, $1/K\sum_{k=1}^{K} \tilde{\phi}_n^{ML,k}(\theta)$ tends to converge to $\hat{\theta}_n^{ML}$ if θ is the true value of the parameter. As a result, $\hat{\theta}_{n,K}^{II}$ may have better finite-sample properties than $\hat{\theta}_n^{ML}$. However, due to the high nonlinearity of the derivative pricing formula, even if $\hat{\theta}_{n,K}^{II}$ has little bias, $P(\hat{\theta}_{n,K}^{II})$ may still be severely biased, sometimes even worse than $\hat{P}_n^{ML}$.

In order to improve the finite-sample properties of $\hat{P}_n^{ML}$, Phillips and Yu (2009) proposed a direct simulation-based method for contingent-claims pricing. They considered the time-series case, where the data generating process of the stock price is described as equation (1). This direct simulation method has the following steps:

(1) We get $\hat{\theta}_n^{ML}$ from the observed data. Inserting $\hat{\theta}_n^{ML}$ into the pricing formula $P(\theta)$, we have $\hat{P}_n^{ML} = P(\hat{\theta}_n^{ML})$;

(2) Given a value of the derivative price p, by $p = P(\theta)$, we get $\theta(p) = P^{-1}(p)$. Here, $P^{-1}(\cdot)$ is the inverse of the pricing formula $P(\theta)$;

(3) For any $\theta(p)$, from equation (1), we can get the simulated data $\tilde{S}^k(p) = \{\tilde{S}_1^k, \tilde{S}_2^k, ..., \tilde{S}_n^k\}$, where $k = 1, 2, ..., K$ is the simulation path. Same as before, in order to calibrate the finite-sample bias, we choose the number of the observations in $\tilde{S}^k(p)$ to be the same as the number of the real data;

(4) From the k-th simulation path, we get the MLE of θ, $\tilde{\phi}_n^{ML,k}(p)$. Then, inserting $\tilde{\phi}_n^{ML,k}(p)$ into the pricing formula, we get $\tilde{P}_n^{ML,k} = P\big(\tilde{\phi}_n^{ML,k}(p)\big)$;

(5) Choose p so that the average behavior of $\tilde{P}_n^{ML,k}$ is matched with $\hat{P}_n^{ML}$, i.e.,

$$\hat{P}_{n,K}^{SM} = \arg\min_p \left\| \hat{P}_n^{ML} - \frac{1}{K}\sum_{k=1}^{K} \tilde{P}_n^{ML,k}(p) \right\|$$

where the binding function is the mean.

Intuitively, whenever bias occurs in $\hat{P}_n^{ML}$ and from whatever source, this bias will also be present in $\tilde{P}_n^{ML,k}(p)$ for the same reason. So, as K goes to infinity, $1/K\sum_{k=1}^{K}\tilde{P}_n^{ML,k}(p)$ tends to converge to $\hat{P}_n^{ML}$ if p is the true value of the option price. As a result, $\hat{P}_{n,K}^{SM}$ approaches $P(\theta)$.

Alternatively, if the median is chosen to be the binding function, the estimator is

$$\hat{P}_{n,K}^{SM} = \arg\min_p \left\| \hat{P}_n^{ML} - \hat{\rho}_{0.5}\tilde{P}_n^{ML,k}(p) \right\|$$

where $\hat{\rho}_{0.5}\tilde{P}_n^{ML,k}(p)$ is the median of $\big\{\tilde{P}_n^{ML,1}(p), \tilde{P}_n^{ML,2}(p), ..., \tilde{P}_n^{ML,K}(p)\big\}$.

2.2.2 Cross-sectional Case

In this section, we assume that the relation between the theoretical derivative prices and the observed prices are described by the following model:

$$\hat{P}_i(\tau_i, X_i) = P_i(\sigma^2; \tau_i, X_i) + \varepsilon_i, \quad \varepsilon_i \sim N(0, \sigma_e^2) \tag{2.5}$$

Here, our observed data are the derivative prices $\hat{P}_i(\tau_i, X_i)$, $i = 1, 2, ..., n$, and they are contingent on the same underlying asset. τ_i is the time to maturity of option i. X_i is the strike price of option i. $P_i(\sigma^2; \tau_i, X_i)$ is the theoretical option price from the Black-Scholes model. Our objective is to price the derivative price $\hat{P}_{n+1}(\tau_{n+1}, X_{n+1})$, given time to maturity τ_{n+1} and strike price X_{n+1}.

In this cross-sectional case, if we use OLS method to estimate σ^2 and σ_e^2, we get

$$\hat{\sigma}^2 = \arg\min_{\sigma^2} \sum_{i=1}^{n} \left(\hat{P}_i(\tau_i, X_i) - P_i(\sigma^2; \tau_i, X_i)\right)^2$$

and

$$\hat{\sigma}_e^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(\hat{P}_i(\tau_i, X_i) - P_i(\hat{\sigma}^2; \tau_i, X_i)\right)^2$$

Insert $\hat{\sigma}^2$ into the derivative pricing formula, then we get the estimate of the option price of time to maturity τ_{n+1} and strike price X_{n+1},

$$\hat{P}_{n+1} = P(\hat{\sigma}^2; \tau_{n+1}, X_{n+1})$$

Because it is a cross-sectional case, we expect little bias from the OLS estimator of σ^2. However, same as we discussed in the previous section, due to the nonlinearity of the derivative pricing formula, $\hat{P}_{n+1}$ may suffer severe bias.

If we using the direct simulation-based method to estimate the option price, it is implemented as follows:

(1) From the observed data, using OLS method, we can get $\hat{\sigma}^2$ and $\hat{\sigma}_e^2$. Then insert $\hat{\sigma}^2$ into the derivative pricing formula, and we get $\hat{P}_{n+1} = P(\hat{\sigma}^2; \tau_{n+1}, X_{n+1})$;

(2) Given a value of the option price p, time to maturity τ_{n+1} and strike price X_{n+1}, by $p = P(\sigma^2; \tau_{n+1}, X_{n+1})$, we get $\sigma^2(p) = P^{-1}(p; \tau_{n+1}, X_{n+1})$ and $\hat{\sigma}_e^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(\hat{P}_i(\tau_i, X_i) - P_i(\sigma^2(p); \tau_i, X_i)\right)^2$. Here, $P^{-1}(\cdot)$ is the inverse of the pricing formula $P(\sigma^2)$;

(3) Given $\sigma^2(p)$, $\{\tau_i, X_i\}_{i=1}^{n}$ and $\hat{\sigma}_e^2$, from equation (5), we get the simulated data $\tilde{P}^k(p) = \left\{\tilde{P}_1^k(p), \tilde{P}_2^k(p), ..., \tilde{P}_n^k(p)\right\}$, where $k = 1, 2, ..., K$ is the simulation path. Same as before, in order to calibrate the finite-sample bias, we choose the number of observations in $\tilde{P}^k(p)$ to be the same as the number of the observed data;

(4) From the k-th simulation path, $\tilde{P}^k(p) = \left\{\tilde{P}^k(p; \tau_1, X_1), \tilde{P}^k(p; \tau_2, X_2), ..., \tilde{P}^k(p; \tau_n, X_n)\right\}$, we get the OLS estimate of σ^2, say, $\tilde{\sigma}_k^2$,

$$\tilde{\sigma}_k^2(p) = \arg\min_{\sigma^2} \sum_{i=1}^{n} \left(\tilde{P}^k(p; \tau_i, X_i) - P\big(\sigma^2(p); \tau_i, X_i\big)\right)^2$$

Then insert $\tilde{\sigma}_k^2$ into $P(\sigma^2; \tau_{n+1}, X_{n+1})$, and we get

$$\tilde{P}_{n+1}^k(p) = P(\tilde{\sigma}_k^2(p); \tau_{n+1}, X_{n+1})$$

(5) Choose p so that the average behavior of $\tilde{P}_{n+1}^k$ is matched with $\hat{P}_{n+1}$, i.e.,

$$\hat{P}_{n+1}^{SM,K} = \arg\min_p \left\| \hat{P}_{n+1} - \frac{1}{K} \sum_{k=1}^{K} \tilde{P}_{n+1}^k(p) \right\|$$

where the binding function is the mean.

Intuitively, whenever bias occurs in $\hat{P}_{n+1}$ and from whatever source, this bias will also be present in $\tilde{P}_{n+1}^k(p)$ for the same reason. So, $1/K \sum_{k=1}^{K} \tilde{P}_{n+1}^k(p)$ tends to converge to $\hat{P}_{n+1}$ if p is the true value of the option price. As a result, $\hat{P}_{n+1}^{SM,K}$ approaches $P(\sigma^2; \tau_{n+1}, X_{n+1})$ as K goes to infinity.

Alternatively, if the median is chosen to be the binding function, the estimator is

$$\hat{P}_{n+1}^{SM,K} = \arg\min_p \left\| \hat{P}_{n+1} - \hat{\rho}_{0.5} \tilde{P}_{n+1}^k(p) \right\|$$

where $\hat{\rho}_{0.5} \tilde{P}_{n+1}^k(p)$ is the median of $\left\{ \tilde{P}_{n+1}^1(p), \tilde{P}_{n+1}^2(p), ..., \tilde{P}_{n+1}^K(p) \right\}$.

3 Monte Carlo Simulation

3.1 Time-series case: Black-Scholes Model

In order to illustrate the finite-sample problems of $\hat{P}_n^{ML}$, we consider the example of estimating the price of a deep-out-of-the-money option in the context of Black-Scholes model. In this section, let $S(t)$ be the underlying stock price at time t. And we assume $S(t)$ follows the geometric Brownian motion process, which is used by Black and Scholes to price European option (Black and Scholes (1973)). The reasons we choose this kind model are that the estimates of the parameter of the underlying asset σ^2 have analytic solutions, and that the pricing formula has a tractable form. We assume the stock prices follow the following process,

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t) \tag{3.1}$$

Let $\{S_t\}_{t=0}^n$ be a sample of time-series observation of stock price $S(t)$ with sampling interval h and $T = nh$. In the Black-Scholes option pricing formula, the only relevant unknown parameter is σ^2. In this case, the log-likelihood function is available. By Itô's lemma, $\ln S$ has the log-normal property when S follows the process in equation (3.1), i.e.,

$$\ln \frac{S(t+1)}{S(t)} \sim N(0, \sigma\sqrt{dt})$$

Hence, the MLE of σ^2 is

$$\begin{aligned}\hat{\sigma}_n^{2,ML} &= \frac{1}{nh}\sum_{t=0}^{n-1}(\ln\frac{S_{t+1}}{S_t} - \frac{1}{n}\sum_{t=0}^{n-1}\ln\frac{S_{t+1}}{S_t})^2 \\ &= \frac{1}{T}\sum_{t=0}^{n-1}(\ln\frac{S_{t+1}}{S_t} - \frac{1}{n}\sum_{t=0}^{n-1}\ln\frac{S_{t+1}}{S_t})^2\end{aligned}$$

and $\hat{P}_n^{ML} = P(\hat{\sigma}_n^{2,ML})$.

We use 250 daily stock return to estimate the price of a European call option price whose time to maturity is 1 month (21 days) and strike price is X. We define the following notations and the parameter values:

S_0: initial stock price,

X: strike prices,

τ: time to maturity,

r: interest rate,

h: sampling interval,

$\hat{\sigma}_n^{2,ML}$: MLE of σ^2 defined as $\frac{1}{T}\sum_{t=0}^{n-1}(\ln\frac{S_{t+1}}{S_t} - \frac{1}{n}\sum_{t=0}^{n-1}\ln\frac{S_{t+1}}{S_t})^2$,

s_n^2: bias-corrected MLE of σ^2 defined as $\frac{n}{n-1}\hat{\sigma}_n^{2,ML}$,

P: Price of a European call option obtained from Black-Scholes pricing formula,

$P(\sigma^2, \tau, X) = S\Phi(d_1) - Xe^{-r\tau}\Phi(d_2)$,

Φ: the cumulative distribution function of standard normal distribution,

$d_1 = \frac{1}{\sigma\sqrt{\tau}}\big(\ln(S_0/X) + (r + 0.5\sigma^2)\tau\big)$,

$d_2 = \frac{1}{\sigma\sqrt{\tau}}\big(\ln(S_0/X) + (r - 0.5\sigma^2)\tau\big)$,

$\hat{P}_n^{SM,1}$: the simulation-based estimator when mean is chosen to be the binding function,

$\hat{P}_n^{SM,2}$: the simulation-based estimator when median is chosen to be the binding function.

The simulated data comes from the model

$$\ln\frac{S(t+1)}{S(t)} \sim N(0, \sigma\sqrt{h})$$

where $S_0 = 100$,

$X = \alpha \cdot S_0\exp(r\tau)$. When $\alpha = 0.95$, the option is in the money; when $\alpha = 1$, the option is at the money; $\alpha = 1.2$, the option is out of the money;

$\tau = \frac{21}{250}$,

$r = 5\%$,

$h = \frac{1}{250}$,

$\sigma^2 = 0.4$.

The experiment is replicated 5000 times. Here are the simulation results:

Table 3.1 In-the-money Option, True value=9.8960, α=0.95

Estimator	mean	bias (in %)	std err	RMSE	median
$\hat{\sigma}_n^{2,ML}$	0.3981	-0.4791	0.0364	0.0365	0.3971
s_n^2	0.3997	-0.0794	0.0366	0.0366	0.3986
$P(\hat{\sigma}_n^{2,ML})$	9.8731	-0.2322	0.3110	0.3119	9.8709
$P(s_n^2)$	9.8867	-0.0943	0.3117	0.3119	9.8845
$\hat{P}_n^{SM,1}$	9.8960	-0.0001	0.3122	0.3122	9.8938
$\hat{P}_n^{SM,2}$	9.8983	0.0224	0.3123	0.3123	9.8960

Table 3.1 shows the simulation results of the in-the-money option, i.e. $\alpha = 0.95$. The strike price of this option is $X = 0.95 \cdot S_0\exp(r\tau)$. By Black-Scholes model, the true value

of this call option is 9.8960. In this case, the percentage bias of the MLE of σ^2 is small, say, -0.4719%. Since it is an in-the-money option, $P(\cdot)$ is not severely nonlinear. $P(\hat{\sigma}_n^{2,ML})$ has little percentage bias, -0.2322%. Moreover, $P(\hat{\sigma}_n^{2,ML})$ has the smallest standard error, 0.3110. s_n^2 reduces the percentage bias to -0.0794%. As a result, the plug-in estimator $P(s_n^2)$ reduces the percentage bias to -0.0943%, but slightly increases the standard error to 0.3117. The bias is further reduced by the direct simulation-based estimator $\hat{P}_n^{SM,1}$ to -0.0001% when mean is chosen to be the binding function. Similarly, the bias is further reduced by the direct simulation-based estimator $\hat{P}_n^{SM,2}$ to 0.0224% when median is chosen to be the binding function. Moreover, $\hat{P}_n^{SM,2}$ is even median unbiased. While both simulation-based estimators reduce the bias over $P(\hat{\sigma}_n^{2,ML})$, they slightly increase the standard error. As a result, these four estimators have similar performance in terms of RMSE.

Table 3.2 At-the-money Option, True value=7.3025, α=1

Estimator	mean	bias (in %)	std err	RMSE	median
$\hat{\sigma}_n^{2,ML}$	0.3981	-0.4791	0.0364	0.0365	0.3971
s_n^2	0.3997	-0.0794	0.0366	0.0366	0.3986
$P(\hat{\sigma}_n^{2,ML})$	7.2774	-0.3440	0.3320	0.3329	7.2756
$P(s_n^2)$	7.2919	-0.1446	0.3327	0.3328	7.2902
$\hat{P}_n^{SM,1}$	7.3025	-0.0000	0.3331	0.3331	7.3007
$\hat{P}_n^{SM,2}$	7.3043	0.0241	0.3332	0.3332	7.3025

Table 3.2 shows the simulation results of the at-the-money option, i.e., $\alpha = 1$. The results are quite similar with the results in Table 1. By Black-Scholes model, the true value of this call option is 7.3025. In this case, the percentage bias of the MLE of σ^2 is small, say, -0.4719%. Since it is an at-the-money option, $P(\cdot)$ is not strongly nonlinear. $P(\hat{\sigma}_n^{2,ML})$ has little percentage bias, -0.3440%. Moreover, $P(\hat{\sigma}_n^{2,ML})$ has the smallest standard error, 0.3320. s_n^2 reduces the percentage bias to -0.0794%. As a result, the plug-in estimator $P(s_n^2)$ reduce the percentage bias to -0.1446%, but slightly increases the standard error to 0.3327. The bias is further reduced by the direct simulation-based estimator $\hat{P}_n^{SM,1}$ to -0.0000% when mean is chosen to be the binding function. Similarly, the bias is further reduced by the direct simulation-based estimator $\hat{P}_n^{SM,2}$ to 0.0241% when median is chosen to be the binding function. While both simulation-based estimators reduce the bias over $P(\hat{\sigma}_n^{2,ML})$, they slightly increase the standard error. So, these four estimators have similar performance in terms of RMSE.

Table 3.3 shows the simulation results of the out-of-the-money option, i.e., $\alpha = 1.2$. The results are also similar with the results in Table 1 and Table 2 except that, the bias

Table 3.3 Out-of-the-money Option, True value=1.6856, α=1.2

Estimator	mean	bias (in %)	std err	RMSE	median
$\hat{\sigma}_n^{2,ML}$	0.3981	-0.4791	0.0364	0.0365	0.3971
s_n^2	0.3997	-0.0794	0.0366	0.0366	0.3986
$P(\hat{\sigma}_n^{2,ML})$	1.6739	-0.6949	0.2210	0.2213	1.6677
$P(s_n^2)$	1.6836	-0.1194	0.2219	0.2219	1.6774
$\hat{P}_n^{SM,1}$	1.6856	-0.0014	0.2225	0.2225	1.6794
$\hat{P}_n^{SM,2}$	1.6918	0.3685	0.2226	0.2227	1.6856

of $P(\hat{\sigma}_n^{2,ML})$ is larger than that of the previous 2 cases. By Black-Scholes model, the true value of this call option is 1.6856. In this case, it is an out-of-the-money option, and $P(\cdot)$ is nonlinear. $P(\hat{\sigma}_n^{2,ML})$ has larger percentage bias, -0.6949%, but $P(\hat{\sigma}_n^{2,ML})$ still has the smallest standard error, 0.2210. s_n^2 reduces the percentage bias to -0.0794%. As a result, the plug-in estimator $P(s_n^2)$ reduce the percentage bias to -0.1194%, but slightly increases the standard error to 0.2219. The bias is further reduced by the direct simulation-based estimator $\hat{P}_n^{SM,1}$ to -0.0014% when mean is chosen to be the binding function. When median is chosen to be the binding function, the median of the estimator of the option price is exact the true value. While both simulation-based estimators reduce the bias over $P(\hat{\sigma}_n^{2,ML})$, they do not significantly increase the standard error or RMSE.

3.2 Cross-Sectional Data of Option Prices

In this section, we consider the situation: at a time spot, we observe many option prices of different time to maturity and different strike prices contingent on the same underlying asset, and we want to estimate another option price contingent on the same underlying asset given its time to maturity and strike price. Our simulated data of call option prices come from the following model

$$\hat{P}_i(\tau_i, X_i) = P_i(\sigma^2; \tau_i, X_i) + \varepsilon_i, \quad \varepsilon_i \sim N(0, \sigma_e^2) \tag{3.2}$$

The notations are the same as last section. We define the following parameter values:

$S_0 = 100$

$r = 0.05$

$\sigma^2 = 0.4$

At first, we fix α, and let τ_i vary. Then $X_i = \alpha \cdot S_0 \cdot \exp(r \cdot \tau_i)$. When $\alpha = 1$, it is an at-the-money option; when $\alpha = 0.95$, it is an in-the-money option; when $\alpha > 1(1.1, 1.2...)$, it is an out-of-the-money option.

$P_i(\sigma^2; \tau_i, X_i) = S_0\Phi(d_1) - X_i \exp(-r \cdot \tau_i)\Phi(d_2)$

$\hat{\sigma}^2$: OLS estimator of σ^2, defined as

$$\hat{\sigma}^2 = \arg\min_{\sigma^2} \sum_{i=1}^{n} \left(\hat{P}_i(\tau_i, X_i) - P_i(\sigma^2; \tau_i, X_i)\right)^2$$

$\hat{P}^{OLS} = P(\hat{\sigma}^2)$

$\hat{P}_n^{SM,1}$: the simulation-based estimator when mean is chosen to be the binding function,

$\hat{P}_n^{SM,2}$: the simulation-based estimator when median is chosen to be the binding function.

The experiment is replicated 5000 times. We take σ_e around 10% of the theoretical option price, which is reasonable. In the first simulation, let $\tau_i = \frac{1}{250}, \frac{2}{250}, \frac{3}{250}, \frac{4}{250}$, and $\tau_{n+1} = \frac{5}{250}$, i.e., we want to estimate the spot price of the option whose time to maturity is 5 days (5/250), given the spot price of the option contingent on the same stock whose time to maturity is 1,2,3,4 days (1/250, 2/250, 3/250, 4/250). The simulation results are shown from Table 4 to Table 8.

Table 3.4 In-the-money option price, $\alpha = 0.95$, True value P=6.5341, $\sigma_e = 0.5$

Estimator	mean	bias (in %)	std err	RMSE	median	median bias (in %)
$\hat{\sigma}^2$	0.4031	0.7630	0.1129	0.1129	0.4001	0.0293
$\hat{P}^{OLS}$	6.5264	-0.1169	0.4158	0.4158	6.5345	0.0066
$\hat{P}^{SM,1}$	6.5339	-0.0025	0.4141	0.4141	6.5421	0.1220
$\hat{P}^{SM,2}$	6.5254	-0.1333	0.4184	0.4184	6.5341	0.0000

Table 3.4 shows the simulation results of the in-the-money option, i.e., $\alpha = 0.95$. In this simulation, the data we observe are the prices of in-the-money option with different time-to-maturity, and our objective is to price another in-the-money option of a given time-to-maturity. By Black-Scholes model, the true value of this call option is 6.5341. We take $\sigma_e = 0.5$. In this case, the percentage bias of the OLS of σ^2 is small, say, 0.7630%. Since it is an in-the-money option, $P(\cdot)$ is not strongly nonlinear. $\hat{P}^{OLS}$ has little percentage bias, -0.1169%. The bias is further reduced by the simulation-based estimator $\hat{P}^{SM,1}$, to -0.0025%. Moreover, the estimator $\hat{P}^{SM,1}$ also decreases the variance, producing an overall gain in RMSE over $\hat{P}^{OLS}$. Another simulation-based estimator $\hat{P}^{SM,2}$ is median-unbiased, but it slightly increases the standard error and RMSE.

Table 3.5 shows the simulation results of the at-the-money option, i.e., $\alpha = 1$. By Black-Scholes model, the true value of this call option is 3.5671. In this case, we take $\sigma_e = 0.3$. The percentage bias of the OLS of σ^2 is small, say, 0.3690%. Since it is an at-the-money

Table 3.5 At-the-money option price, $\alpha = 1$, True value P=3.5671, $\sigma_e = 0.3$

Estimator	mean	bias (in %)	std err	RMSE	median	median bias (in %)
$\hat{\sigma}^2$	0.4015	0.3690	0.0481	0.0482	0.3998	-0.0391
$\hat{P}^{OLS}$	3.5672	0.0032	0.2144	0.2144	3.5664	-0.0195
$\hat{P}^{SM,1}$	3.5671	-0.0000	0.2144	0.2144	3.5662	-0.0228
$\hat{P}^{SM,2}$	3.5670	0.0227	0.2144	0.2144	3.5671	0

option, $P(\cdot)$ is not strongly nonlinear. $\hat{P}^{OLS}$ has little percentage bias, 0.0032%. The bias is further reduced by the simulation-based estimator $\hat{P}^{SM,1}$, which is almost unbiased. If median is taken as the binding function, the simulation-based estimator $\hat{P}^{SM,2}$ is median-unbiased. Moreover, both of the simulation-based estimators, $\hat{P}^{SM,1}$ and $\hat{P}^{SM,2}$ have the similar variance and RMSE as $\hat{P}^{OLS}$.

Table 3.6 Out-of-the-money option price, $\alpha = 1.1$, True value P=0.6883, $\sigma_e = 0.07$

Estimator	mean	bias (in %)	std err	RMSE	median	median bias (in %)
$\hat{\sigma}^2$	0.3997	-0.0756	0.0274	0.0274	0.4002	0.0391
$\hat{P}^{OLS}$	0.6876	-0.0905	0.0724	0.0724	0.6887	0.0601
$\hat{P}^{SM,1}$	0.6883	0.0020	0.0723	0.0723	0.6893	0.1504
$\hat{P}^{SM,2}$	0.6871	-0.1628	0.0724	0.0724	0.6883	0

Table 3.6 shows the simulation results of the out-of-the-money option when $\alpha = 1.1$. By Black-Scholes model, the true value of this call option is 0.6883. So, we take $\sigma_e = 0.07$. In this case, the percentage bias of the OLS of σ^2 is only -0.0756%. As a result, $\hat{P}^{OLS}$ also has little percentage bias, -0.0905%. The bias is further reduced by the simulation-based estimator $\hat{P}^{SM,1}$, to 0.0020%. Meanwhile, it slightly reduces the variance, producing smaller RMSE. Another simulation-based estimator $\hat{P}^{SM,2}$ is almost median-unbiased.

Table 3.7 Out-of-the-money option price, $\alpha = 1.2$, True value P=0.0749, $\sigma_e = 0.007$

Estimator	mean	bias (in %)	std err	RMSE	median	median bias (in %)
$\hat{\sigma}^2$	0.3988	-0.2913	0.0209	0.0210	0.4002	0.0488
$\hat{P}^{OLS}$	0.0748	-0.2395	0.0125	0.0125	0.0751	0.1594
$\hat{P}^{SM,1}$	0.0750	0.0169	0.0125	0.0125	0.0752	0.3975
$\hat{P}^{SM,2}$	0.0746	-0.4194	0.0126	0.0126	0.0749	0

Table 3.7 shows the simulation results of the out-of-the-money option when $\alpha = 1.2$. By Black-Scholes model, the true value of this call option is 0.0749. Here, we take $\sigma_e = 0.007$. In this case, the percentage bias of the OLS of σ^2 is small, say, -0.2913%. Since it is a

severe out-of-the-money option, $P(\cdot)$ is strongly nonlinear. $\hat{P}^{OLS}$ suffer larger percentage bias than before, -0.2395%. The bias is reduced by the simulation-based estimator $\hat{P}^{SM,1}$ to 0.0169%. Moreover, when median is chosen be to the binding function, the simulation-based estimator $\hat{P}^{SM,2}$ is even median-unbiased. Meanwhile, both of the simulation-based estimators do not increase the variance significantly.

Table 3.8 Out-of-the-money option price, $\alpha = 1.3$, True value P=0.0049, $\sigma_e = 0.0003$

Estimator	mean	bias (in %)	std err	RMSE	median	median bias (in %)
$\hat{\sigma}^2$	0.3987	-0.3299	0.0151	0.0152	0.4002	0.0391
$\hat{P}^{OLS}$	0.0049	-0.3527	0.0010	0.0010	0.0049	0.2187
$\hat{P}^{SM,1}$	0.0049	0.0309	0.0010	0.0010	0.0049	0.5685
$\hat{P}^{SM,2}$	0.0049	-0.6018	0.0010	0.0010	0.0049	0

Table 3.8 shows the simulation results of the out-of-the-money option when $\alpha = 1.3$. By Black-Scholes model, the true value of this call option is 0.0049. Here, we take $\sigma_e = 0.0003$. In this case, the percentage bias of the OLS of σ^2 is a littile bit larger, say, -0.3299%. As a result, $\hat{P}^{OLS}$ suffer larger percentage bias than before, -0.3527%. The bias is reduced by the simulation-based estimator $\hat{P}^{SM,1}$ to 0.0309%. Moreover, when median is chosen be to the binding function, the simulation-based estimator $\hat{P}^{SM,2}$ is even median-unbiased. Meanwhile, both of the simulation-based estimators do not increase the variance or RMSE over those of $\hat{P}^{OLS}$.

In the second simulation, we change the value of time-to-maturity. Let $\tau_i = \frac{21}{250}, \frac{42}{250}, \frac{63}{250}, \frac{84}{250}$, and $\tau_{n+1} = \frac{126}{250}$, i.e., we want to estimate the spot price of the option whose time to maturity is 6 months (126/250), given the spot price of the option whose time to maturity is 1,2,3,4 months (21/250, 42/250, 63/250, 84/250). The results are shown from Table 3.9 to Table 3.14.

Table 3.9 In-the-money option price, $\alpha = 0.95$, True value P=19.9300, $\sigma_e = 1.5$

Estimator	mean	bias (in %)	std err	RMSE	median	median bias (in %)
$\hat{\sigma}^2$	0.4019	0.4839	0.0554	0.0555	0.3999	-0.0195
$\hat{P}^{OLS}$	19.9289	-0.0056	1.1709	1.1709	19.9284	-0.0083
$\hat{P}^{SM,1}$	19.9300	0.0001	1.1708	1.1708	19.9295	-0.0027
$\hat{P}^{SM,2}$	19.9311	0.0053	1.1715	1.1715	19.9300	0

From these tables, we find out: (1) As α increases, the true value of the option price decreases. Then the value we take for σ_e decreases. As a result, the percentage bias of

Table 3.10 At-the-money option price, $\alpha = 1$, True value P=17.7631, $\sigma_e = 1.5$

Estimator	mean	bias (in %)	std err	RMSE	median	median bias (in %)
$\hat{\sigma}^2$	0.4018	0.4595	0.0532	0.0532	0.3998	-0.0391
$\hat{P}^{OLS}$	17.7628	-0.0019	1.1601	1.1601	17.7597	-0.0192
$\hat{P}^{SM,1}$	17.7631	-0.0000	1.1602	1.1602	17.7601	-0.0173
$\hat{P}^{SM,2}$	17.7663	0.0181	1.1601	1.1601	17.7631	0

Table 3.11 Out-of-the-money option price, $\alpha = 1.1$, True value P=14.0626, $\sigma_e = 1$

Estimator	mean	bias (in %)	std err	RMSE	median	median bias (in %)
$\hat{\sigma}^2$	0.4007	0.1873	0.0356	0.0356	0.4000	0
$\hat{P}^{OLS}$	14.0615	-0.0077	0.7979	0.7979	14.0626	0
$\hat{P}^{SM,1}$	14.0626	0.0001	0.7978	0.7978	14.0636	0.0077
$\hat{P}^{SM,2}$	14.0620	-0.0039	0.7984	0.7984	14.0626	0

Table 3.12 Out-of-the-money option price, $\alpha = 1.2$, True value P=11.0967, $\sigma_e = 0.8$

Estimator	mean	bias (in %)	std err	RMSE	median	median bias (in %)
$\hat{\sigma}^2$	0.4004	0.1058	0.0312	0.0312	0.3998	-0.0391
$\hat{P}^{OLS}$	11.0941	-0.0233	0.6883	0.6883	11.0932	-0.0310
$\hat{P}^{SM,1}$	11.0967	0.0003	0.6880	0.6880	11.0958	-0.0077
$\hat{P}^{SM,2}$	11.0984	0.0153	0.6885	0.6885	11.0967	0

Table 3.13 Out-of-the-money option price, $\alpha = 1.3$, True value P=8.7388, $\sigma_e = 0.5$

Estimator	mean	bias (in %)	std err	RMSE	median	median bias (in %)
$\hat{\sigma}^2$	0.4001	0.0299	0.0226	0.0226	0.4001	0.0195
$\hat{P}^{OLS}$	8.7366	-0.0254	0.4736	0.4736	8.7405	0.0188
$\hat{P}^{SM,1}$	8.7389	0.0003	0.4734	0.4734	8.7427	0.0441
$\hat{P}^{SM,2}$	8.7347	-0.0475	0.4742	0.4742	8.7388	0

Table 3.14 Out-of-the-money option price, $\alpha = 1.4$, True value P=6.8752, $\sigma_e = 0.5$

Estimator	mean	bias (in %)	std err	RMSE	median	median bias (in %)
$\hat{\sigma}^2$	0.4000	-0.0003	0.0270	0.0270	0.4004	0.0977
$\hat{P}^{OLS}$	6.8708	-0.0631	0.5260	0.5261	6.8828	0.1108
$\hat{P}^{SM,1}$	6.8752	0.0012	0.5254	0.5254	6.8871	0.1738
$\hat{P}^{SM,2}$	6.8638	-0.1658	0.5264	0.5264	6.8752	0

OLS estimator of σ^2, $\hat{\sigma}^2$, decrease. However, the percentage bias of $\hat{P}^{OLS}$ increases due to the nonlinearity of the out-of-the-money option price formula. (2) The percentage bias can be further reduced over that of $\hat{P}^{OLS}$ by the simulation-based estimator $\hat{P}^{SM,1}$. When median is chosen to be the binding function, the simulation-based estimator $\hat{P}^{SM,2}$ is always median-unbiased. (3)Both of the simulation-based estimators do not significantly increase the variance or RMSE over those of $\hat{P}^{OLS}$. In some cases, $\hat{P}^{SM,1}$ even produces smaller variance and RMSE.

In the third simulation, we keep τ fixed, let α vary. Now, the model is,

$$\hat{P}_i(\tau, X_i) = P_i(\sigma^2; \tau, X_i) + \varepsilon_i, \quad \varepsilon_i \sim N(0, \sigma_e^2) \tag{3.3}$$

Similar as before, we define the following notations and parameter values.

$\alpha_i = 0.95, 1, 1.1$, and $\alpha_{n+1} = 1.2$

$S_0 = 100$

$r = 0.05$

$\sigma^2 = 0.4$

$X_i = \alpha_i \cdot S_0 \cdot \exp(r \cdot \tau),$

$P_i(\sigma^2; \tau_i, X_i) = S_0\phi(d_1) - X_i \exp(-r \cdot \tau_i)\phi(d_2)$

Here, for any given τ, the data we observe are the option prices of time-to-maturity τ with different α. We want to estimate the price of the option of the same time-to-maturity with $\alpha = 1.2$, given the price of the option of $\alpha = 0.95, 1, 1.1$. This experiment is replicated 5000 times. Same as before, we take σ_e around 10% of the theoretical option price. The simulation results are shown from Table 3.15 to Table 3.17.

Table 3.15 $\tau = 21/250$**, True value P=1.6856,** $\sigma_e = 1$

Estimator	mean	bias (in %)	std err	RMSE	median	median bias (in %)
$\hat{\sigma}^2$	0.4006	0.1441	0.0333	0.0333	0.4000	0
$\hat{P}^{OLS}$	1.6890	0.2039	0.2020	0.2020	1.6856	0
$\hat{P}^{SM,1}$	1.6856	-0.0014	0.2024	0.2024	1.6822	-0.2042
$\hat{P}^{SM,2}$	1.6888	0.1927	0.2019	0.2019	1.6856	0

Table 3.15 shows the simulation results of the option of time-to-maturity 1 month (21/250). By Black-Scholes model, the true value of this call option is 1.6856. Here, we take $\sigma_e = 1$. The percentage bias of the OLS estimator of σ^2 is 0.1441%. $\hat{P}^{OLS}$ has little percentage bias, say 0.2039%. The bias is further reduced by the simulation-based estimator $\hat{P}^{SM,1}$ to -0.0014%. When median is taken as the binding function, the simulation-based estimator $\hat{P}^{SM,2}$ is median-unbiased. Moreover, both of the simulation-based estimators do not significantly increase the variance or RMSE over $\hat{P}^{OLS}$.

Table 3.16 and Table 3.17 show the simulation results of the option of time-to-maturity

Table 3.16 $\tau = 63/250$**, True value P=6.0749,** $\sigma_e = 1$

Estimator	mean	bias (in %)	std err	RMSE	median	median bias (in %)
$\hat{\sigma}^2$	0.4008	0.2004	0.0373	0.0373	0.4001	0.0195
$\hat{P}^{OLS}$	6.0778	0.0475	0.5412	0.5412	6.0761	0.0187
$\hat{P}^{SM,1}$	6.0749	-0.0008	0.5416	0.5416	6.0732	-0.0289
$\hat{P}^{SM,2}$	6.0767	0.0284	0.5417	0.5417	6.0749	0

Table 3.17 $\tau = 126/250$**, True value P=11.0967,** $\sigma_e = 1.5$

Estimator	mean	bias (in %)	std err	RMSE	median	median bias (in %)
$\hat{\sigma}^2$	0.4010	0.2383	0.0397	0.0397	0.4001	0.0195
$\hat{P}^{OLS}$	11.0985	0.0166	0.8730	0.8730	11.0984	0.0155
$\hat{P}^{SM,1}$	11.0966	-0.0005	0.8735	0.8735	11.0965	-0.0011
$\hat{P}^{SM,2}$	11.0964	-0.0024	0.8729	0.8729	11.0967	0

3 month (63/250)and half a year (126/250) respectively. The results are quite similar as that of Table 15. We find out that the simulation-based estimator $\hat{P}^{SM,1}$ can always reduce the percentage bias of $\hat{P}^{OLS}$. If the median is chosen to be the binding function, the simulation-based estimator $\hat{P}^{SM,2}$ is median-unbiased. Moreover, both of the simulation-based estimators do not significantly increase the variance or RMSE over $\hat{P}^{OLS}$.

4 Conclusion and Further Studies

In this paper, we extend the simulation-based estimation method proposed by Phillips and Yu (2009) to the cross-sectional case. In order to illustrate their finite-sample properties, we conduct Monte-Carlo simulations of this simulation-based method to both the time-series model and the cross-sectional model of the option prices. The simulation results show that the proposed simulation-based estimator can always reduce the percentage bias over the respective MLE and OLS estimator. Meanwhile, the simulation-based estimator does not significantly increase the variance or RMSE over their correspondent MLE and OLS estimator. The findings are consistent with Phillips and Yu (2009) of the time-series case. However, this paper does not consider the problem of misspecification. In further studies, we can analyze how this simulation-based estimator will perform when the model is misspecified.

Reference

AÏT-SAHALIA, Y. (1999): "Transition Densities for Interest Rate and Other Nonlinear Diffusions," *Journal of Finance*, 54, 1361-1395

BUTLER, J. S, AND SCHACHTER, B. (1986): "Unbiased Estimation of the Black/Scholes Formula," *Journal of Financial Economics*, 15, 341-357

DUFFIE, D., AND K. J. SINGLETON (2003): "Simulated Moments Estimation of Markov Models of Asset Prices," *Econometrica*, 61, 929-952

GOURIÉROUX, C., P. C. B. PHILLIPS, AND J. YU: "Indirect inference for dynamic panel models," *Journal of Econometrics*, forthcoming

INGERSOLL, J. E. (1976): "A Theoretical and Empirical Investigation of the Dual Purpose Funds:: An Application of Contingent-claims Analysis," *Journal of Financial Economics*, 3, 83-123

KNIGHT, J. L., AND S. E. SATCHELL (1997): "Existence of Unbiased Estimators of the Black/Scholes Option Price, Other Derivatives, and Hedge Ratios," *Econometric Theory*, 13, 791-807

MACKINNON, J. G., AND A. A. SMITH (1998): "Approximate bias correction in econometrics," *Journal of Econometrics*, 85, 205-230

PHILLIPS, P. C. B., AND J. YU (2005): "Jackknifing Bond Option Prices," *Review of Financial Studies*, 18, 707-742

PHILLIPS, P. C. B., AND J. YU (2009): "Simulation-Based Estimation of Contingent-Claims Prices," *Review of Financial Studies*, 22, 3669-3705

ZEHNA, P. W. (1966): "Invariance of Maximum Likelihood Estimators," *The Annals of Mathematical Statistics*, 37, 744

Acknowledgement

I am deeply indebted to Professor Jun Yu for guidance and encouragement. I thank Professor Liangjun Su and Professor Sainan Jin for helpful comments.